Broadcast Ephemeris Generation by Curve Fitting to Satellite Positions

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1. Ephemeris parameters

$$\boldsymbol{p}_{eph}(t_{oe}) = (a, e, i_0, \Omega_0, \omega, M_0, \Delta n, \dot{I}, \dot{\Omega}, C_{us}, C_{uc}, C_{rs}, C_{rc}, C_{is}, C_{ic})^T$$

2. Ephemeris to satellite position (IS-GPS-200F Table 20-IV)

$$t_k = t - t_{oe}$$
$$M = M_0 + \left(\sqrt{\frac{\mu}{a^3}} + \Delta n\right) t_k$$
$$M = E - e \sin E$$
$$\alpha = \frac{\sqrt{1-e^2} \sin E}{\cos E - e}$$
$$\phi = \arctan \alpha + \omega$$
$$\delta u = C_{us} \sin 2\phi + C_{uc} \cos 2\phi$$
$$\delta r = C_{rs} \sin 2\phi + C_{rc} \cos 2\phi$$
$$\delta i = C_{is} \sin 2\phi + C_{ic} \cos 2\phi$$
$$u = \phi + \delta u$$
$$r = a(1 - e \cos E) + \delta r$$
$$i = i_0 + \delta i + \dot{I}\, t_k$$
$$\Omega = \Omega_0 + (\dot{\Omega} - \omega_e)\, t_k - \omega_e\, t_{oe}$$
$$\boldsymbol{r}_{eph}(t) = r \begin{pmatrix} \cos u \cos\Omega - \sin u \cos i \sin\Omega \\ \cos u \sin\Omega + \sin u \cos i \cos\Omega \\ \sin u \sin i \end{pmatrix}$$

3. Partial derivatives of satellite positions wrt ephemeris parameters

$$\frac{\partial M}{\partial a} = -\frac{3}{2}\sqrt{\frac{\mu}{a^5}} t_k$$
$$\frac{\partial M}{\partial \Delta n} = t_k$$
$$\frac{\partial E}{\partial e} - \sin E - e \cos E \frac{\partial E}{\partial e} = 0$$
$$\frac{\partial E}{\partial e} = \frac{\sin E}{1 - e \cos E}$$
$$\frac{\partial E}{\partial M} - e \cos E \frac{\partial E}{\partial M} = 1$$
$$\frac{\partial E}{\partial M_0} = \frac{\partial E}{\partial M} = \frac{1}{1 - e \cos E}$$

$$\frac{\partial E}{\partial \Delta n} = \frac{\partial E}{\partial M}\frac{\partial M}{\partial \Delta n} = \frac{t_k}{1 - e\cos E}$$

$$\frac{\partial \alpha}{\partial E} = \frac{\sqrt{1-e^2}(1-e\cos E)}{(\cos E - e)^2}$$

$$\frac{\partial \alpha}{\partial e} = \frac{\sqrt{1-e^2}}{\cos E - e}\left\{-\frac{e\sin E}{1-e^2} + \frac{\sin E}{\cos E - e}\left(1 + \sin E\frac{\partial E}{\partial e}\right) + \cos E\frac{\partial E}{\partial e}\right\}$$

$$\frac{\partial \phi}{\partial \alpha} = \frac{1}{1+\alpha^2} = \left(\frac{\cos E - e}{1 - e\cos E}\right)^2$$

$$\frac{\partial \phi}{\partial E} = \frac{\partial \phi}{\partial \alpha}\frac{\partial \alpha}{\partial E} = \frac{\sqrt{1-e^2}}{1-e\cos E}$$

$$\frac{\partial \phi}{\partial a} = \frac{\partial \phi}{\partial E}\frac{\partial E}{\partial a} = \frac{\partial \phi}{\partial E}\frac{\partial E}{\partial M}\frac{\partial M}{\partial a}$$

$$\frac{\partial \phi}{\partial e} = \frac{\partial \phi}{\partial \alpha}\frac{\partial \alpha}{\partial e}$$

$$\frac{\partial \phi}{\partial M_0} = \frac{\partial \phi}{\partial E}\frac{\partial E}{\partial M_0}$$

$$\frac{\partial \phi}{\partial \Delta n} = \frac{\partial \phi}{\partial E}\frac{\partial E}{\partial \Delta n}$$

$$\frac{\partial u}{\partial a} = \left\{1 + 2\left(C_{us}\cos 2\phi - C_{uc}\sin 2\phi\right)\right\}\frac{\partial \phi}{\partial a}$$

$$\frac{\partial u}{\partial e} = \left\{1 + 2\left(C_{us}\cos 2\phi - C_{uc}\sin 2\phi\right)\right\}\frac{\partial \phi}{\partial e}$$

$$\frac{\partial u}{\partial \omega} = 1 + 2\left(C_{us}\cos 2\phi - C_{uc}\sin 2\phi\right)$$

$$\frac{\partial u}{\partial M_0} = \left\{1 + 2\left(C_{us}\cos 2\phi - C_{uc}\sin 2\phi\right)\right\}\frac{\partial \phi}{\partial M_0}$$

$$\frac{\partial u}{\partial \Delta n} = \left\{1 + 2\left(C_{us}\cos 2\phi - C_{uc}\sin 2\phi\right)\right\}\frac{\partial \phi}{\partial \Delta n}$$

$$\frac{\partial r}{\partial a} = 1 - e\cos E + ae\sin E\frac{\partial E}{\partial M}\frac{\partial M}{\partial a} + 2\left(C_{rs}\cos 2\phi - C_{rc}\sin 2\phi\right)\frac{\partial \phi}{\partial a}$$

$$\frac{\partial r}{\partial e} = a\left(-\cos E + e\sin E\frac{\partial E}{\partial e}\right) + 2\left(C_{rs}\cos 2\phi - C_{rc}\sin 2\phi\right)\frac{\partial \phi}{\partial e}$$

$$\frac{\partial r}{\partial \omega} = 2\left(C_{rs}\cos 2\phi - C_{rc}\sin 2\phi\right)$$

$$\frac{\partial r}{\partial M_0} = ae\sin E\frac{\partial E}{\partial M_0} + 2\left(C_{rs}\cos 2\phi - C_{rc}\sin 2\phi\right)\frac{\partial \phi}{\partial M_0}$$

$$\frac{\partial r}{\partial \Delta n} = ae\sin E\frac{\partial E}{\partial \Delta n} + 2\left(C_{rs}\cos 2\phi - C_{rc}\sin 2\phi\right)\frac{\partial \phi}{\partial \Delta n}$$

$$\frac{\partial i}{\partial a} = 2\left(C_{is}\cos 2\phi - C_{ic}\sin 2\phi\right)\frac{\partial \phi}{\partial a}$$

$$\frac{\partial i}{\partial e} = 2\left(C_{is}\cos 2\phi - C_{ic}\sin 2\phi\right)\frac{\partial \phi}{\partial e}$$

$$\frac{\partial i}{\partial \omega} = 2\left(C_{is}\cos 2\phi - C_{ic}\sin 2\phi\right)$$

$$\frac{\partial i}{\partial M_0} = 2\left(C_{is}\cos 2\phi - C_{ic}\sin 2\phi\right)\frac{\partial \phi}{\partial M_0}$$

$$\frac{\partial i}{\partial \Delta n} = 2\left(C_{is}\cos 2\phi - C_{ic}\sin 2\phi\right)\frac{\partial \phi}{\partial \Delta n}$$

$$X = \cos u \, \cos \Omega - \sin u \, \cos i \, \sin \Omega$$
$$Y = \cos u \, \sin \Omega + \sin u \, \cos i \, \cos \Omega$$
$$V = -\sin u \, \cos \Omega - \cos u \, \cos i \, \sin \Omega$$
$$W = -\sin u \, \sin \Omega + \cos u \, \cos i \, \cos \Omega$$
$$\frac{\partial x}{\partial a} = X \frac{\partial r}{\partial a} + r\left(V \frac{\partial u}{\partial a} + \sin u \, \sin i \, \sin \Omega \frac{\partial i}{\partial a} \right)$$
$$\frac{\partial x}{\partial e} = X \frac{\partial r}{\partial e} + r\left(V \frac{\partial u}{\partial e} + \sin u \, \sin i \, \sin \Omega \frac{\partial i}{\partial e} \right)$$
$$\frac{\partial x}{\partial i_0} = r \sin u \, \sin i \, \sin \Omega$$
$$\frac{\partial x}{\partial \Omega_0} = -r\, Y$$
$$\frac{\partial x}{\partial \omega} = X \frac{\partial r}{\partial \omega} + r\left(V \frac{\partial u}{\partial \omega} + \sin u \, \sin i \, \sin \Omega \frac{\partial i}{\partial \omega} \right)$$
$$\frac{\partial x}{\partial M_0} = X \frac{\partial r}{\partial M_0} + r\left(V \frac{\partial u}{\partial M_0} + \sin u \, \sin i \, \sin \Omega \frac{\partial i}{\partial M_0} \right)$$
$$\frac{\partial x}{\partial \Delta n} = X \frac{\partial r}{\partial \Delta n} + r\left(V \frac{\partial u}{\partial \Delta n} + \sin u \, \sin i \, \sin \Omega \frac{\partial i}{\partial \Delta n} \right)$$
$$\frac{\partial x}{\partial \dot{I}} = r \sin u \, \sin i \, \sin \Omega \, t_k$$
$$\frac{\partial x}{\partial \dot{\Omega}} = -r\, Y \, t_k$$
$$\frac{\partial x}{\partial C_{us}} = r V \sin 2\phi$$
$$\frac{\partial x}{\partial C_{uc}} = r V \cos 2\phi$$
$$\frac{\partial x}{\partial C_{rs}} = X \sin 2\phi$$
$$\frac{\partial x}{\partial C_{rc}} = X \cos 2\phi$$
$$\frac{\partial x}{\partial C_{is}} = r \sin u \, \sin i \, \sin \Omega \, \sin 2\phi$$
$$\frac{\partial x}{\partial C_{ic}} = r \sin u \, \sin i \, \sin \Omega \, \cos 2\phi$$
$$\frac{\partial y}{\partial a} = Y \frac{\partial r}{\partial a} + r\left(W \frac{\partial u}{\partial a} - \sin u \, \sin i \, \cos \Omega \frac{\partial i}{\partial a} \right)$$
$$\frac{\partial y}{\partial e} = Y \frac{\partial r}{\partial e} + r\left(W \frac{\partial u}{\partial e} - \sin u \, \sin i \, \cos \Omega \frac{\partial i}{\partial e} \right)$$
$$\frac{\partial y}{\partial i_0} = -r \sin u \, \sin i \, \cos \Omega$$
$$\frac{\partial y}{\partial \Omega_0} = r\, X$$
$$\frac{\partial y}{\partial \omega} = Y \frac{\partial r}{\partial \omega} + r\left(W \frac{\partial u}{\partial \omega} - \sin u \, \sin i \, \cos \Omega \frac{\partial i}{\partial \omega} \right)$$
$$\frac{\partial y}{\partial M_0} = Y \frac{\partial r}{\partial M_0} + r\left(W \frac{\partial u}{\partial M_0} - \sin u \, \sin i \, \cos \Omega \frac{\partial i}{\partial M_0} \right)$$

$$\frac{\partial y}{\partial \Delta n} = Y \frac{\partial r}{\partial \Delta n} + r \left(W \frac{\partial u}{\partial \Delta n} - \sin u \, \sin i \, \cos \Omega \frac{\partial i}{\partial \Delta n} \right)$$

$$\frac{\partial y}{\partial \dot{I}} = -r \sin u \, \sin i \, \cos \Omega \, t_k$$

$$\frac{\partial y}{\partial \dot{\Omega}} = r \, X \, t_k$$

$$\frac{\partial y}{\partial C_{us}} = r W \sin 2\phi$$

$$\frac{\partial y}{\partial C_{uc}} = r W \cos 2\phi$$

$$\frac{\partial y}{\partial C_{rs}} = Y \sin 2\phi$$

$$\frac{\partial y}{\partial C_{rc}} = Y \cos 2\phi$$

$$\frac{\partial y}{\partial C_{is}} = -r \sin u \, \sin i \, \cos \Omega \, \sin 2\phi$$

$$\frac{\partial y}{\partial C_{ic}} = -r \sin u \, \sin i \, \cos \Omega \, \cos 2\phi$$

$$\frac{\partial z}{\partial a} = \sin u \, \sin i \frac{\partial r}{\partial a} + r \left(\cos u \, \sin i \frac{\partial u}{\partial a} + \sin u \, \cos i \frac{\partial i}{\partial a} \right)$$

$$\frac{\partial z}{\partial e} = \sin u \, \sin i \frac{\partial r}{\partial e} + r \left(\cos u \, \sin i \frac{\partial u}{\partial e} + \sin u \, \cos i \, \frac{\partial i}{\partial e} \right)$$

$$\frac{\partial z}{\partial i_0} = r \sin u \, \cos i$$

$$\frac{\partial z}{\partial \Omega_0} = 0$$

$$\frac{\partial z}{\partial \omega} = \sin u \, \sin i \frac{\partial r}{\partial \omega} + r \left(\cos u \, \sin i \frac{\partial u}{\partial \omega} + \sin u \, \cos i \frac{\partial i}{\partial \omega} \right)$$

$$\frac{\partial z}{\partial M_0} = \sin u \, \sin i \frac{\partial r}{\partial M_0} + r \left(\cos u \, \sin i \frac{\partial u}{\partial M_0} + \sin u \, \sin i \frac{\partial i}{\partial M_0} \right)$$

$$\frac{\partial z}{\partial \Delta n} = \sin u \, \sin i \frac{\partial r}{\partial \Delta n} + r \left(\cos u \, \sin i \frac{\partial u}{\partial \Delta n} + \sin u \, \sin i \frac{\partial i}{\partial \Delta n} \right)$$

$$\frac{\partial z}{\partial \dot{I}} = r \sin u \, \cos i \, t_k$$

$$\frac{\partial z}{\partial \dot{\Omega}} = 0$$

$$\frac{\partial z}{\partial C_{us}} = r \cos u \, \sin i \, \sin 2\phi$$

$$\frac{\partial z}{\partial C_{uc}} = r \cos u \, \sin i \, \cos 2\phi$$

$$\frac{\partial z}{\partial C_{rs}} = \sin u \, \sin i \, \sin 2\phi$$

$$\frac{\partial z}{\partial C_{rc}} = \sin u \, \sin i \, \cos 2\phi$$

$$\frac{\partial z}{\partial C_{is}} = r \sin u \, \cos i \, \sin 2\phi$$

$$\frac{\partial z}{\partial C_{ic}} = r \sin u \, \cos i \, \cos 2\phi$$

$$\boldsymbol{H}(t) = \frac{\partial \boldsymbol{r}_{eph}(t)}{\partial \boldsymbol{p}_{eph}(t_{oe})} =$$

$$\begin{pmatrix} \frac{\partial x}{\partial a} & \frac{\partial x}{\partial e} & \frac{\partial x}{\partial i_0} & \frac{\partial x}{\partial \Omega_0} & \frac{\partial x}{\partial \omega} & \frac{\partial x}{\partial M_0} & \frac{\partial x}{\partial \Delta n} & \frac{\partial x}{\partial \dot{I}} & \frac{\partial x}{\partial \dot{\Omega}} & \frac{\partial x}{\partial C_{us}} & \frac{\partial x}{\partial C_{uc}} & \frac{\partial x}{\partial C_{rs}} & \frac{\partial x}{\partial C_{rc}} & \frac{\partial x}{\partial C_{is}} & \frac{\partial x}{\partial C_{ic}} \\ \frac{\partial y}{\partial a} & \frac{\partial y}{\partial e} & \frac{\partial y}{\partial i_0} & \frac{\partial y}{\partial \Omega_0} & \frac{\partial y}{\partial \omega} & \frac{\partial y}{\partial M_0} & \frac{\partial y}{\partial \Delta n} & \frac{\partial y}{\partial \dot{I}} & \frac{\partial y}{\partial \dot{\Omega}} & \frac{\partial y}{\partial C_{us}} & \frac{\partial y}{\partial C_{uc}} & \frac{\partial y}{\partial C_{rs}} & \frac{\partial y}{\partial C_{rc}} & \frac{\partial y}{\partial C_{is}} & \frac{\partial y}{\partial C_{ic}} \\ \frac{\partial z}{\partial a} & \frac{\partial z}{\partial e} & \frac{\partial z}{\partial i_0} & \frac{\partial z}{\partial \Omega_0} & \frac{\partial z}{\partial \omega} & \frac{\partial z}{\partial M_0} & \frac{\partial z}{\partial \Delta n} & \frac{\partial z}{\partial \dot{I}} & \frac{\partial z}{\partial \dot{\Omega}} & \frac{\partial z}{\partial C_{us}} & \frac{\partial z}{\partial C_{uc}} & \frac{\partial z}{\partial C_{rs}} & \frac{\partial z}{\partial C_{rc}} & \frac{\partial z}{\partial C_{is}} & \frac{\partial z}{\partial C_{ic}} \end{pmatrix}$$

4. Least square estimation of ephemeris parameters

$$\boldsymbol{p}_{eph,0}(t_{oe}) = (a_{toe}, e_{toe}, i_{toe}, \Omega_{toe}, \omega_{toe}, M_{toe}, 0,0,0,0,0,0,0,0,0)^T$$

$a_{toe}, e_{toe}, i_{toe}, \Omega_{toe}, \omega_{toe}, M_{toe}$: osculating orbital elements at TOE

(1) Gauss-Newton

do loop *until convergence*:

$$\boldsymbol{v}_i = \begin{pmatrix} \boldsymbol{r}(t_1) - \boldsymbol{r}_{eph,i}(t_1) \\ \boldsymbol{r}(t_2) - \boldsymbol{r}_{eph,i}(t_2) \\ \vdots \\ \boldsymbol{r}(t_n) - \boldsymbol{r}_{eph,i}(t_n) \end{pmatrix}, \quad \boldsymbol{H}_i = \begin{pmatrix} \boldsymbol{H}(t_1) \\ \boldsymbol{H}(t_2) \\ \vdots \\ \boldsymbol{H}(t_n) \end{pmatrix}$$

$$\boldsymbol{p}_{eph,i+1}(t_{oe}) = \boldsymbol{p}_{eph,i}(t_{oe}) + (\boldsymbol{H}_i^T \boldsymbol{H}_i)^{-1} \boldsymbol{H}_i^T \boldsymbol{v}_i$$

(2) Levenberg-Marquardt

do loop *until convergence*:

$$\boldsymbol{v}_i = \begin{pmatrix} \boldsymbol{r}(t_1) - \boldsymbol{r}_{eph,i}(t_1) \\ \boldsymbol{r}(t_2) - \boldsymbol{r}_{eph,i}(t_2) \\ \vdots \\ \boldsymbol{r}(t_n) - \boldsymbol{r}_{eph,i}(t_n) \end{pmatrix}, \quad \boldsymbol{H}_i = \begin{pmatrix} \boldsymbol{H}(t_1) \\ \boldsymbol{H}(t_2) \\ \vdots \\ \boldsymbol{H}(t_n) \end{pmatrix}$$

$$s_i = \boldsymbol{v}_i^T \boldsymbol{v}_i$$

if $s_i < s_{i-1}$:

$\lambda \leftarrow \lambda / 10$

$$\boldsymbol{p}_{eph,i+1}(t_{oe}) = \boldsymbol{p}_{eph,i}(t_{oe}) + (\boldsymbol{H}_i^T \boldsymbol{H}_i + \lambda \boldsymbol{I})^{-1} \boldsymbol{H}_i^T \boldsymbol{v}_i$$

else :

$\lambda \leftarrow \lambda * 10$

$$\boldsymbol{p}_{eph,i+1}(t_{oe}) = \boldsymbol{p}_{eph,i-1}(t_{oe}) + (\boldsymbol{H}_{i-1}^T \boldsymbol{H}_{i-1} + \lambda \boldsymbol{I})^{-1} \boldsymbol{H}_{i-1}^T \boldsymbol{v}_{i-1}$$