For JAXA R&D

PPP - Models, Algorithms and Implementations (5)

Tokyo Univ. of Marine Science and Technology (TUMSAT)

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2019-12-06 @Tokyo, Japan

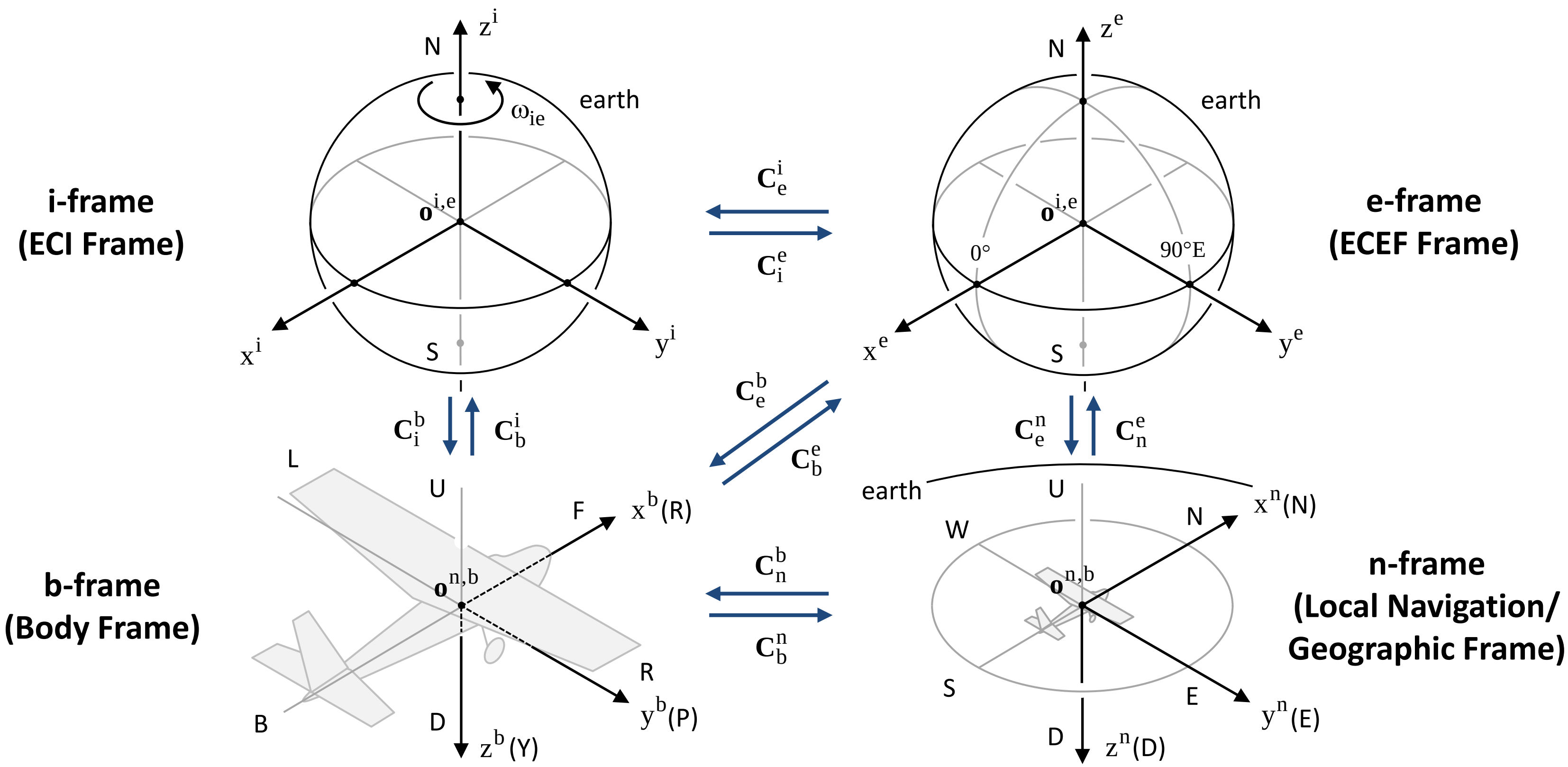
PPP - Models, Algorithms and Implementation

1. 2019-10-04 **PPP models**
   geometric range, ionosphere, troposphere, antenna PCV, earth tides, wind-up, relativity, biases, coordinates
2. 2019-10-18 **PPP algorithms**
   SPP, LSQ, GN, EKF, noise-model, RAIM/QC, LAPACK/BLAS
3. 2019-11-01 **PPP data handling**
   LC, interpolation, slip detection, RINEX, SP3, ANTEX, RTCM, CSSR
4. 2019-11-22 **PPP-AR**
   UPD/FCB, EWL/WL/NL, ILS, LAMBDA, TCAR, PAR, validation
5. 2019-12-06 **INS integration**
   INS sensors, Inertial navigation, INS integration
6. 2019-12-20 **POD of satellites**
   orbit element, orbit model, reduced-dynamic, ECI-ECEF transformation, precession/nutation, EOP

(1.5 h / session)

Inertial Navigation

Coordinate Frames



Frame Transformations

$$\mathbf{C}_i^e = \mathbf{R}_z(\omega_{ie}(t - t_0)), \mathbf{C}_e^n = \mathbf{R}_y(-\pi/2 - L_{en})\mathbf{R}_z(\lambda_{en}), \mathbf{C}_n^b = \mathbf{R}_x(\phi_{nb})\mathbf{R}_y(\theta_{nb})\mathbf{R}_z(\psi_{nb}),$$

$$\mathbf{C}_e^b = \mathbf{C}_n^b\mathbf{C}_e^n,\ \mathbf{C}_i^b = \mathbf{C}_e^b\mathbf{C}_i^e,\ \mathbf{C}_e^i = \mathbf{C}_i^{eT},\ \mathbf{C}_n^e = \mathbf{C}_e^{nT},\ \mathbf{C}_b^n = \mathbf{C}_n^{bT},\ \mathbf{C}_b^e = \mathbf{C}_e^{bT},\ \mathbf{C}_b^i = \mathbf{C}_i^{bT}$$

$\mathbf{C}_\beta^\alpha$: Transformation from β-frame to α-frame
$\mathbf{o}^{i,e}$: i- and e-frame origin (earth mass center)
$\mathbf{o}^{n,b}$: n- and b-frame origin

ω_{ie} : Earth rotation rate (rad/s)
L_{en} : Latitude of n- and b-frame origin (rad)
λ_{en} : Longitude of n- and b-frame origin (rad)

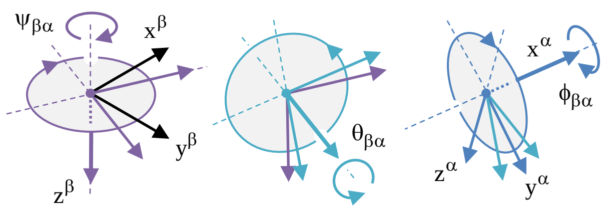
ϕ_{nb} : Roll angle of b- wrt n-frame (rad)
θ_{nb} : Pitch angle of b- wrt n-frame (rad)
ψ_{nb} : Yaw angle of b- wrt n-frame (rad)

Attitudes and Rotations

(1) Euler Angles/Attitude

$$\boldsymbol{\psi}_{\beta\alpha} = (\phi_{\beta\alpha}, \theta_{\beta\alpha}, \psi_{\beta\alpha})^T$$

$\phi_{\beta\alpha}, \theta_{\beta\alpha}, \psi_{\beta\alpha}$: Roll, Pitch and Yaw angles of α-frame wrt β-frame



(1) to (2) : $\mathbf{C}_\beta^\alpha = \mathbf{R}_x(\phi_{\beta\alpha})\mathbf{R}_y(\theta_{\beta\alpha})\mathbf{R}_z(\psi_{\beta\alpha})$

(1) to (2) →

← **(2) to (1)**

(2) to (3) ↙

(4) to (2) ↑

(2) DCM/CTM

$$\mathbf{C}_\beta^\alpha = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}$$

$\mathbf{C}_\beta^\alpha$: Transformation from β-frame to α-frame

(2) to (1) : $\boldsymbol{\psi}_{\beta\alpha} = \begin{pmatrix} \mathrm{ATAN2}(c_{23}, c_{33}) \\ -\arcsin(c_{13}) \\ \mathrm{ATAN2}(c_{12}, c_{11}) \end{pmatrix}$

(2) to (3) :

$$\mu_{\beta\alpha} = \arccos(1/2(c_{11} + c_{22} + c_{33} - 1))$$

$$\mathbf{e}_{\beta\alpha}^{\alpha/\beta} = \frac{1}{2\sin\mu_{\beta\alpha}} \begin{pmatrix} c_{23} - c_{32} \\ c_{31} - c_{13} \\ c_{12} - c_{21} \end{pmatrix}$$

(3) Rotation Vector

$$\boldsymbol{\rho}_{\beta\alpha}^{\alpha/\beta} = \mu_{\beta\alpha}\mathbf{e}_{\beta\alpha}^{\alpha/\beta}$$

$\mu_{\beta\alpha}$: Rotation angle of α-frame wrt β-frame

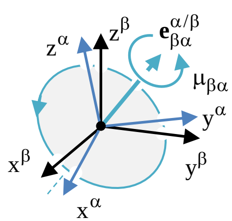
$\mathbf{e}_{\beta\alpha}^{\alpha/\beta}$: Unit vector of rotation axis

(3) to (4) →

(4) Quaternion

$$\mathbf{q}_\beta^\alpha = (q_0, q_1, q_2, q_3)^T$$

$$(q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1)$$



(3) to (4) :

$$\mathbf{q}_\beta^\alpha = \begin{pmatrix} \cos(\mu_{\beta\alpha}/2) \\ \mathbf{e}_{\beta\alpha}^{\alpha/\beta}\sin(\mu_{\beta\alpha}/2) \end{pmatrix}$$

Note: The $\mathbf{e}_{\beta\alpha}^{\alpha/\beta}$ is one of the eigenvectors of the $\mathbf{C}_\beta^\alpha$

(4) to (2) :

$$\mathbf{C}_\beta^\alpha = \begin{pmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1q_2 + q_3q_0) & 2(q_1q_3 - q_2q_0) \\ 2(q_1q_2 - q_3q_0) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2q_3 + q_1q_0) \\ 2(q_1q_3 + q_2q_0) & 2(q_2q_3 - q_1q_0) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{pmatrix}$$

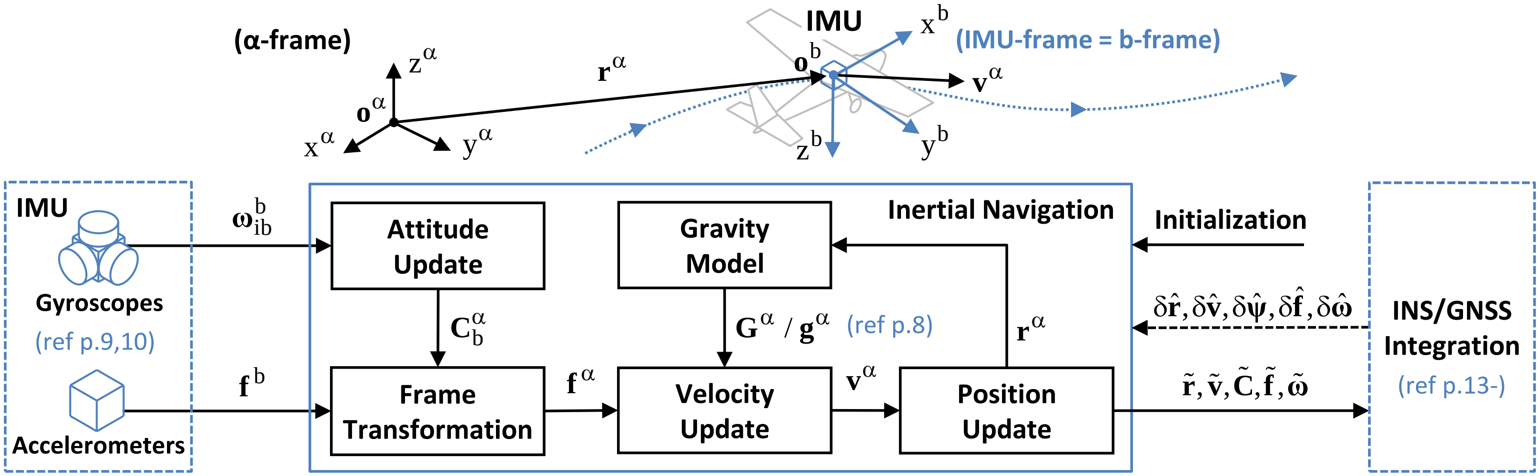
$\mathbf{x}_{\beta\alpha}^\gamma$: $\mathbf{x}$ of α-frame wrt β-frame expressed in γ-frame axes

$$\mathbf{R}_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}, \mathbf{R}_y(\theta) = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}, \mathbf{R}_z(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \boldsymbol{\Omega} = (\boldsymbol{\omega}\times) = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}$$

: Skew-symmetric matrix of angular rate vector $\boldsymbol{\omega} = (\omega_x, \omega_y, \omega_z)^T$

Inertial Navigation

Strapdown Inertial Navigation



Attitude, velocity and position update in i-frame

$$\mathbf{C}_b^i(t+\tau) = \mathbf{C}_b^i(t)\exp\left(((\boldsymbol{\omega}_{ib}^b - \delta\boldsymbol{\omega})\times)\tau\right) \quad \text{(ref p.7)}$$

$$\mathbf{f}^i = \frac{1}{2}\left(\mathbf{C}_b^i(t) + \mathbf{C}_b^i(t+\tau)\right)(\mathbf{f}^b - \delta\mathbf{f})$$

$$\mathbf{v}^i(t+\tau) = \mathbf{v}^i(t) + \left(\underbrace{\mathbf{f}^i + \mathbf{G}^i(\mathbf{r}^i(t))}_{\text{(acceleration)}}\right)\tau$$

$$\mathbf{r}^i(t+\tau) = \mathbf{r}^i(t) + \frac{1}{2}\left(\mathbf{v}^i(t) + \mathbf{v}^i(t+\tau)\right)\tau$$

Attitude, velocity and position update in e-frame

$$\mathbf{C}_b^e(t+\tau) = \mathbf{C}_b^e(t)\exp\left(((\boldsymbol{\omega}_{ib}^b - \delta\boldsymbol{\omega} \underbrace{- \mathbf{C}_b^e(t)^T\boldsymbol{\omega}_{ie}^e}_{\text{(earth rotation)}})\times)\tau\right)$$

$$\mathbf{f}^e = \frac{1}{2}\left(\mathbf{C}_b^e(t) + \mathbf{C}_b^e(t+\tau)\right)(\mathbf{f}^b - \delta\mathbf{f})$$

$$\mathbf{v}^e(t+\tau) = \mathbf{v}^e(t) + \left(\mathbf{f}^e + \mathbf{g}^e(\mathbf{r}^e(t)) \underbrace{- 2\boldsymbol{\omega}_{ie}^e \times \mathbf{v}^e(t)}_{\text{(Coriolis force)}}\right)\tau$$

$$\mathbf{r}^e(t+\tau) = \mathbf{r}^e(t) + \frac{1}{2}\left(\mathbf{v}^e(t) + \mathbf{v}^e(t+\tau)\right)\tau$$

- $\mathbf{r}^\alpha$: Position in α-frame(m)
- $\mathbf{v}^\alpha$: Velocity in α-frame (m/s)
- $\mathbf{C}_b^\alpha$: Transformation from b- to α-frame
- $\boldsymbol{\omega}_{ib}^b$: Angular rate measurement (rad/s)
- $\mathbf{f}^b$: Specific force measurement (m/s^2)
- $\mathbf{f}^\alpha$: Specific force in α-frame (m/s^2)
- $\delta\mathbf{f}$: Accelerometer bias (m/s2)
- $\delta\boldsymbol{\omega}$: Gyro bias (rad/s)
- $\mathbf{G}^\alpha$: Gravitational acceleration (m/s^2)
- $\mathbf{g}^\alpha$: Acceleration due to gravity (m/s^2)
- $\boldsymbol{\omega}_{ie}^e$: Earth rotation vector (rad/s)
- τ : Update interval (s)

Attitude Update

Update DCM by Angular Rate

$$
\begin{aligned}
\mathbf{C}_b^\alpha(t+\tau) &\approx \mathbf{C}_b^\alpha(t)\exp\left(\int_t^{t+\tau}(\boldsymbol{\omega}_{\alpha b}^b(t)\times)\,du\right)\\
&\approx \mathbf{C}_b^\alpha(t)\exp\left((\boldsymbol{\omega}_{\alpha b}^b(t)\times)\tau\right)\\
&= \mathbf{C}_b^\alpha(t)\exp(\boldsymbol{\sigma}_{\alpha b}^b\times)\\
&= \mathbf{C}_b^\alpha(t)\left(\mathbf{I}+\frac{\sin\sigma}{\sigma}(\boldsymbol{\sigma}_{\alpha b}^b\times)+\frac{1-\cos\sigma}{\sigma^2}(\boldsymbol{\sigma}_{\alpha b}^b\times)^2\right)\\
&\approx \mathbf{C}_b^\alpha(t)\left(\mathbf{I}+a_1(\boldsymbol{\sigma}_{\alpha b}^b\times)+a_2(\boldsymbol{\sigma}_{\alpha b}^b\times)^2\right)
\end{aligned}
$$

$\boldsymbol{\omega}$: Angular rate (rad/s)
$\boldsymbol{\sigma}$: Angle increments (rad)

$(\sigma=|\boldsymbol{\sigma}|)$

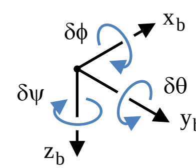
Note: Attitude updates should be performed at the IMU sampling rate to minimize integration errors

Drift Error in Computed Attitude by Approximation [1]

Order	Algorithm	Attitude Drift Error (deg/h)	
		$\sigma_{max}=5.73°$	$\sigma_{max}=2.86°$
1	$a_1=1, a_2=0$	6870	1720
2	$a_1=1, a_2=1/2$	3430	860
3	$a_1=1-\sigma^2/6, a_2=1/2$	7	0.4
4	$a_1=1-\sigma^2/6, a_2=1/2-\sigma^2/24$	1.7	0.1

Derivative of DCM

$$
\begin{aligned}
\dot{\mathbf{C}}_b^\alpha(t) &= \lim_{\delta t\to 0}\left(\frac{\mathbf{C}_b^\alpha(t+\delta t)-\mathbf{C}_b^\alpha(t)}{\delta t}\right)\\
&= \lim_{\delta t\to 0}\left(\frac{\mathbf{C}_b^\alpha(t)\mathbf{C}_{b(t+\delta t)}^{b(t)}-\mathbf{C}_b^\alpha(t)}{\delta t}\right)\\
&= \lim_{\delta t\to 0}\left(\frac{\mathbf{C}_b^\alpha(t)(\mathbf{I}+\delta\boldsymbol{\Psi})-\mathbf{C}_b^\alpha(t)}{\delta t}\right)\\
&= \mathbf{C}_b^\alpha(t)\lim_{\delta t\to 0}\left(\frac{\delta\boldsymbol{\Psi}}{\delta t}\right)\\
&= \mathbf{C}_b^\alpha(t)(\boldsymbol{\omega}_{\alpha b}^b(t)\times)=\mathbf{C}_b^\alpha(t)\boldsymbol{\Omega}_{\alpha b}^b(t)
\end{aligned}
$$



$$
\delta\boldsymbol{\Psi}=\begin{pmatrix}0 & -\delta\psi & \delta\theta\\ \delta\psi & 0 & -\delta\phi\\ -\delta\theta & \delta\phi & 0\end{pmatrix}
$$

$$
\boldsymbol{\Omega}=\begin{pmatrix}0 & -\omega_z & \omega_y\\ \omega_z & 0 & -\omega_x\\ -\omega_y & \omega_x & 0\end{pmatrix}
$$

Skew-Symmetric Matrix Exponential

$$
\begin{aligned}
\exp(\boldsymbol{\sigma}\times) &= \sum_{n=0}^{\infty}\frac{(\boldsymbol{\sigma}\times)^n}{n!}=\mathbf{I}+(\boldsymbol{\sigma}\times)+\frac{(\boldsymbol{\sigma}\times)^2}{2!}+\frac{(\boldsymbol{\sigma}\times)^3}{3!}+\frac{(\boldsymbol{\sigma}\times)^4}{4!}+\ldots\\
&= \mathbf{I}+\left(1-\frac{\sigma^2}{3!}+\frac{\sigma^4}{5!}-\ldots\right)(\boldsymbol{\sigma}\times)+\left(\frac{1}{2!}-\frac{\sigma^2}{4!}+\frac{\sigma^4}{6!}-\ldots\right)(\boldsymbol{\sigma}\times)^2\\
&= \mathbf{I}+\frac{\sin\sigma}{\sigma}(\boldsymbol{\sigma}\times)+\frac{1-\cos\sigma}{\sigma^2}(\boldsymbol{\sigma}\times)^2
\end{aligned}
$$

$$
(\boldsymbol{\sigma}\times)=\begin{pmatrix}0 & -\sigma_z & \sigma_y\\ \sigma_z & 0 & -\sigma_x\\ -\sigma_y & \sigma_x & 0\end{pmatrix},\quad (\boldsymbol{\sigma}\times)^2=\begin{pmatrix}-\sigma_y^2-\sigma_z^2 & \sigma_x\sigma_y & \sigma_x\sigma_z\\ \sigma_x\sigma_y & -\sigma_x^2-\sigma_z^2 & \sigma_y\sigma_z\\ \sigma_x\sigma_z & \sigma_y\sigma_z & -\sigma_y^2-\sigma_z^2\end{pmatrix}
$$

$$
(\boldsymbol{\sigma}\times)^3=-\sigma^2(\boldsymbol{\sigma}\times),\ (\boldsymbol{\sigma}\times)^4=-\sigma^2(\boldsymbol{\sigma}\times)^2 \qquad (\sigma^2=\sigma_x^2+\sigma_y^2+\sigma_z^2)
$$

[1] D. H. Titterton and J. L. Weston, Strapdown Inertial Navigation Technology, Second Edition, AIAA and The Institution of Engineering and Technology, 2004

Gravity Model

Acceleration by Gravity

$$\mathbf{G}^e(\mathbf{r}^e) = \nabla U(\mathbf{r}^e) = \nabla\left\{\frac{\mu}{r}\left(1+\sum_{n=2}^{nmax}\left(\frac{a}{r}\right)^n\sum_{m=0}^{n}(\bar{C}_{nm}\cos m\lambda + \bar{S}_{nm}\sin m\lambda)\bar{P}_{nm}(\sin L')\right)\right\}$$

$$\approx -\frac{\mu}{r^3}\left(\mathbf{r}^e + \frac{3}{2}J_2\left(\frac{a}{r}\right)^2\begin{pmatrix}(1-5(z^e/r)^2)x^e\\(1-5(z^e/r)^2)y^e\\(3-5(z^e/r)^2)z^e\end{pmatrix}\right)$$

(approx. error
= ~ 10^{-4} m/s²
= ~ 10 ug)

$$\mathbf{r}^e = (x^e, y^e, z^e)^T$$
$$\mathbf{r}^i = (x^i, y^i, z^i)^T$$
$$r = |\mathbf{r}^e| = |\mathbf{r}^i|$$

$$\mathbf{G}^i(\mathbf{r}^i) \approx -\frac{\mu}{r^3}\left(\mathbf{r}^i + \frac{3}{2}J_2\left(\frac{a}{r}\right)^2\begin{pmatrix}(1-5(z^i/r)^2)x^i\\(1-5(z^i/r)^2)y^i\\(3-5(z^i/r)^2)z^i\end{pmatrix}\right)$$

$$\mathbf{g}^e(\mathbf{r}^e) = \mathbf{G}^e(\mathbf{r}^e) \underline{-\mathbf{\Omega}^e_{ie}\mathbf{\Omega}^e_{ie}\mathbf{r}^e}$$
$$= \mathbf{G}^e(\mathbf{r}^e) \underline{+\omega_{ie}{}^2(x^e, y^e, 0)^T}$$

(centrifugal accel = ~ 0.03 m/s²)

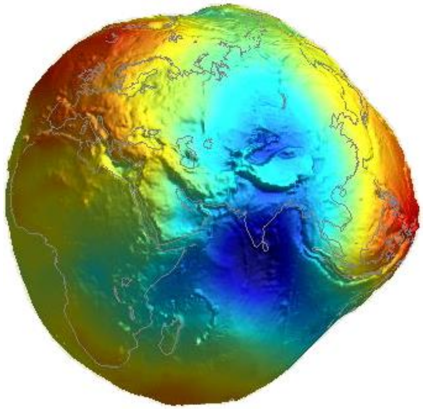
WGS84 Constants [1]

$$a = 6378137.0\,\mathrm{m}$$
$$f = 1/298.257223563$$
$$\mu = 3.986004418\times10^{14}\,\mathrm{m^3/s^2}$$
$$\omega_{ie} = 7.292115\times10^{-5}\,\mathrm{rad/s}$$
$$J_2 = 1.08262982\times10^{-3}$$

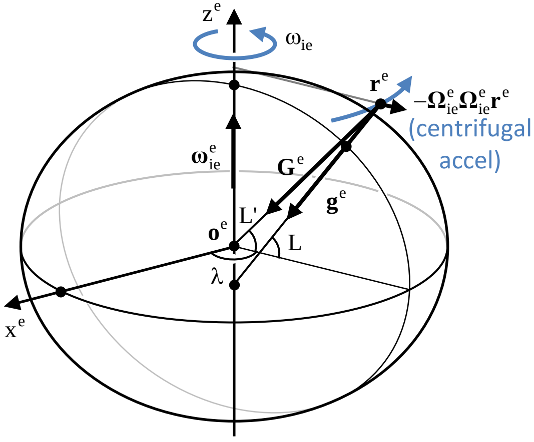
Partial Derivatives wrt Position

$$\frac{\partial \mathbf{G}^i}{\partial \mathbf{r}^i} \approx -\frac{\mu}{r^3}\left(\mathbf{I} - \frac{3}{r^2}\mathbf{r}^i\mathbf{r}^{iT}\right), \frac{\partial \mathbf{g}^e}{\partial \mathbf{r}^e} \approx -\frac{\mu}{r^3}\left(\mathbf{I} - \frac{3}{r^2}\mathbf{r}^e\mathbf{r}^{eT}\right) - \mathbf{\Omega}^e_{ie}\mathbf{\Omega}^e_{ie}$$

- $\mathbf{G}^\alpha(\mathbf{r}^\alpha)$: Gravitational acceleration (m/s²)
- U : Earth gravitational potential
- μ : Earth gravitational constant (m³/s²)
- a : Semi-major axis of ellipsoid (m)
- $\bar{P}_{nm}$: Associated Legendre function
- $\bar{C}_{nm}, \bar{S}_{nm}$: Gravitational coefficients
- $\mathbf{g}^e(\mathbf{r}^e)$: Acceleration due to Gravity (m/s²)
- L : Geodetic latitude (rad)
- L' : Geocentric latitude (rad)
- λ : Longitude (rad)
- ω_{ie} : Earth rotation rate (rad/s)
- $\boldsymbol{\omega}^e_{ie}$: Earth rotation vector (rad/s)



Earth Gravitational Potential



$$\boldsymbol{\omega}^e_{ie} = (0, 0, \omega_{ie})^T,\ \mathbf{\Omega}^e_{ie} = (\boldsymbol{\omega}^e_{ie}\times)$$

[1] NGA.STND.0036_1.0.0_WGS84, World Geodetic System 1984 - Its definition and relationships with local geodetic systems, version 1.0.0, 2014

IMU (Inertial Measurement Unit)

Gyroscopes

Mechanical: RIG (rate integrating gyro), DTG (dynamically tuned gyro), ...

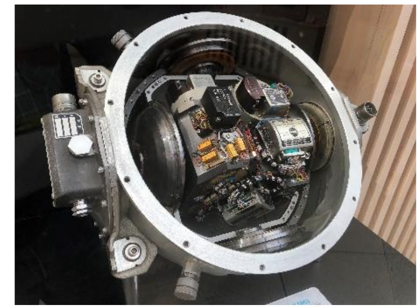
Vibratory: HRG (hemispherical resonator gyro), Wine grass resonator, ...

Optical: RLG (ring laser gyro), FOG (fiber optic gyro), ...

Accelerometers

Mechanical: Force-feedback pendulous accelerometer, Vibrating beam, ...

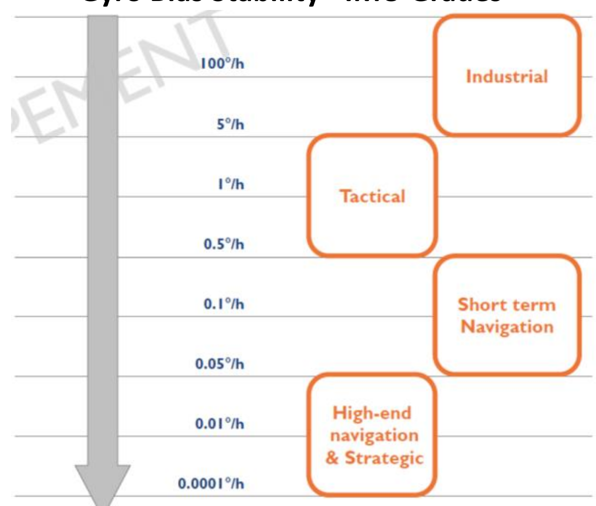
Solid-state: Silicon sensors, Optical accelerometers, ...



Apollo IMU [2]

IMU Vendors [1]

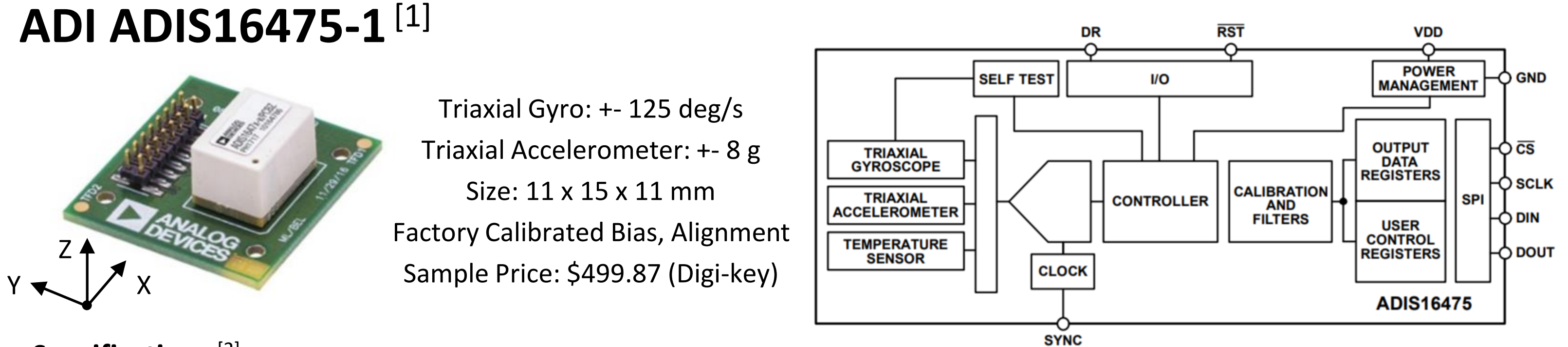
Gyro Bias Stability - IMU Grades [1]



[1] Yole Development, High-end gyroscopes, accelerometers and IMUs for defense, aerospace & industrial, 2015
[2] https://en.wikipedia.org/wiki/Inertial_measurement_unit

MEMS IMU

ADI ADIS16475-1 [1]



Triaxial Gyro: +- 125 deg/s

Triaxial Accelerometer: +- 8 g

Size: 11 x 15 x 11 mm

Factory Calibrated Bias, Alignment

Sample Price: $499.87 (Digi-key)

Specifications [2]

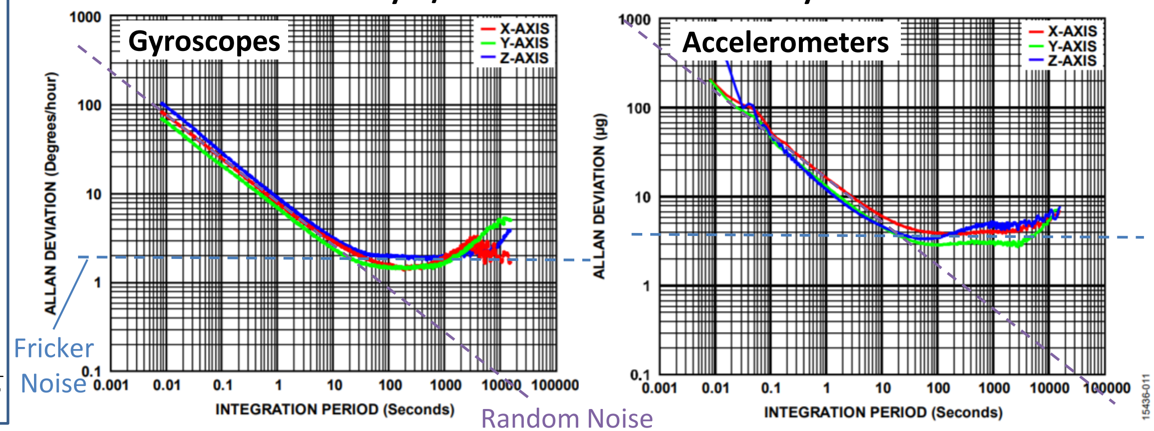
Gyroscopes

Misalignment Error	:	+- 0.1°
Nonlinearity	:	0.2 % FS
In-Run Bias Stability	:	2 °/hr
Angular Random Walk	:	0.15 $°/\sqrt{\mathrm{hr}}$

Accelerometers

Misalignment Error	:	+- 0.05°
Nonlinearity (+- 2g)	:	0.25 % FS
In-Run Bias Stability	:	3.6 μg
Velocity Random Walk	:	0.012 $\mathrm{m/s}/\sqrt{\mathrm{hr}}$

Gyro/Accelerometer Bias Stability [2]



Expected Position Error due to IMU Noise (t = 60 s, v = 60 km/h)

Cross-Track: $\underbrace{(2\times\pi/180/3600}_{\text{gyro stability (rad/s)}}\times\underbrace{60}_{\text{time (s)}}/2)\times\underbrace{(60\times10^3/3600\times60)}_{\text{distance (m)}}=0.291\ (\mathrm{m})$

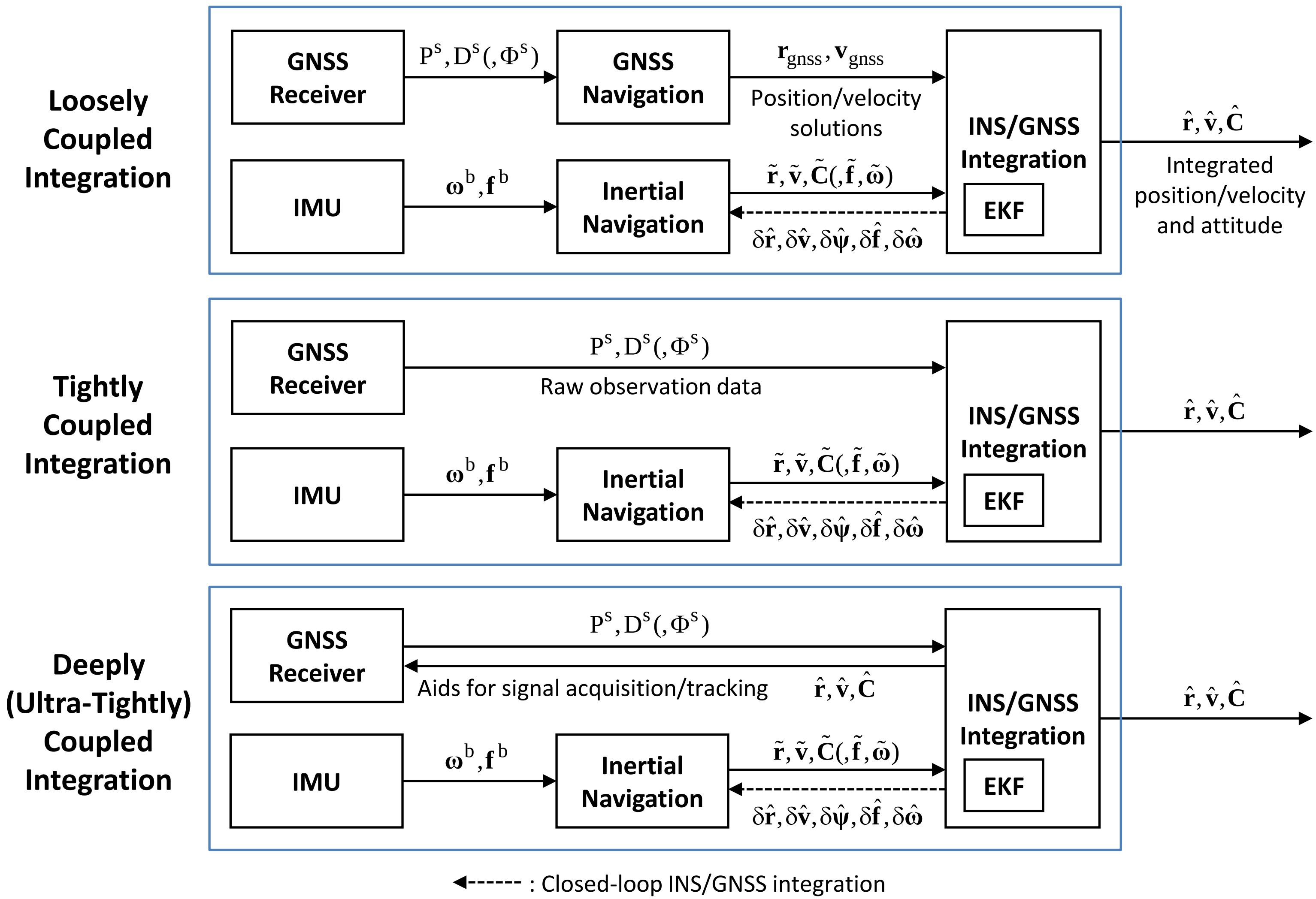
Along-Track: $\underbrace{(3.6\times10^{-6}\times9.8)}_{\text{accelerometer stability (m/s}^2\text{)}}\times\underbrace{60^2}_{\text{time (s)}}/2=0.064\ (\mathrm{m})$

[1] https://www.analog.com/en/products/adis16475.html

[2] Analog Devices, Precision, Miniature MEMs IMU ADIS16475 Data Sheet, Rev.C, 2019

INS/GNSS Integration

Integration Architectures



GNSS Velocity Solution

Parameters and Measurements

$$\mathbf{x} = (\mathbf{v}_r{}^T, c\dot{dt}_r)^T, \mathbf{y} = (-\lambda D_r^1, -\lambda D_r^2, -\lambda D_r^3, \ldots, -\lambda D_r^m)^T$$

Non-linear LSE

$$\mathbf{y} = \mathbf{h}(\mathbf{x}) + \boldsymbol{\varepsilon}$$

$$\mathbf{h}(\mathbf{x}) = \begin{pmatrix} \dot{\rho}_r^1 + c\dot{t}_r - c\dot{T}^1(t^1) \\ \dot{\rho}_r^2 + c\dot{t}_r - c\dot{T}^2(t^2) \\ \dot{\rho}_r^3 + c\dot{t}_r - c\dot{T}^3(t^3) \\ \vdots \\ \dot{\rho}_r^m + c\dot{t}_r - c\dot{T}^m(t^m) \end{pmatrix}, \quad \mathbf{H} = \begin{pmatrix} -\mathbf{e}_r^{1T} & 1 \\ -\mathbf{e}_r^{2T} & 1 \\ -\mathbf{e}_r^{3T} & 1 \\ \vdots & \vdots \\ -\mathbf{e}_r^{mT} & 1 \end{pmatrix}$$

$$\dot{\rho}_r^s = \mathbf{e}_r^{sT}(\mathbf{v}^s(t^s) - \mathbf{v}_r) + \delta\dot{\rho}$$

$$t^s = \bar{t}_r - P_r^s/c - dT^s(t^s), \; \mathbf{v}^s(t^s) \approx \frac{\mathbf{r}^s(t^s + \Delta t) - \mathbf{r}^s(t^s)}{\Delta t}$$

$$\delta\dot{\rho} = \frac{\omega_{ie}(v_y^s x_r + y^s v_{x,r} - v_x^s y_r - x^s v_{y,r})}{c}$$

$$\hat{\mathbf{x}}_0 = \mathbf{0}$$

$$\hat{\mathbf{x}}_{i+1} = \hat{\mathbf{x}}_i + (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T(\mathbf{y} - \mathbf{h}(\hat{\mathbf{x}}_i)) \qquad \mathbf{Q}_{\hat{v}} = \sigma_D^2(\mathbf{H}^T\mathbf{H})^{-1}$$

$$\hat{\mathbf{x}} = \lim_{i\to\infty} \hat{\mathbf{x}}_i = (\hat{\mathbf{v}}_r{}^T, c\hat{\dot{dt}}_r)^T$$

$\mathbf{r}_r$: Receiver position by SPP (ECEF) (m)
$\mathbf{v}_r$: Receiver velocity (ECEF) (m/s)
$\dot{dt}_r$: Receiver clock drift (s/s)
P_r^s : Pseudorange (m)
D_r^s : Doppler frequency (Hz)
λ : Carrier wavelength (m/cyc)
$\mathbf{r}^s(t^s)$: Satellite position (ECEF) (m)
$\mathbf{v}^s(t^s)$: Satellite velocity (ECEF) (m/s)
$dT^s(t^s)$: Satellite clock bias (s)
$d\dot{T}^s(t^s)$: Satellite clock drift (s/s)
$\mathbf{e}_r^s$: LOS (line-of-sight) vector
$\bar{t}_r$: Signal reception time by receiver clock (s)
t^s : Signal transmission time (s)
$\dot{\rho}_r^s$: Range rate (m/s)
$\delta\dot{\rho}$: Sagnac effect correction for range rate (m/s)
σ_D : Std-dev of doppler noise (m/s)

$$\mathbf{r}_r = (x_r, y_r, z_r)^T, \; \mathbf{v}_r = (v_{x,r}, v_{y,r}, v_{z,r})^T$$

$$\mathbf{r}^s(t^s) = (x^s, y^s, z^s)^T, \; \mathbf{v}^s(t^s) = (v_x^s, v_y^s, v_z^s)^T$$

$\hat{v}_r, \mathbf{Q}_{\hat{v}}$: GNSS velocity solution (ECEF) (m/s) and VC-matrix

INS/GNSS Integration (1/3)

State Vector and their Propagation in e-frame

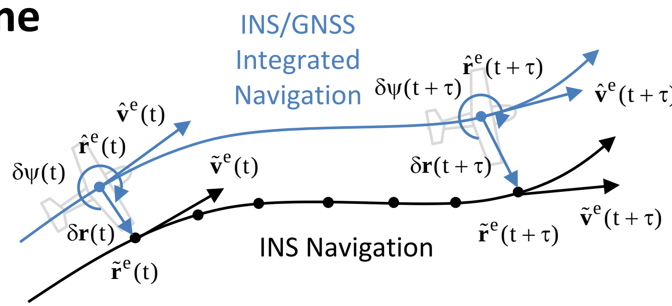
$$\mathbf{x}_{\text{ins}} = (\delta\mathbf{r}^T, \delta\mathbf{v}^T, \delta\boldsymbol{\psi}^T, \delta\mathbf{f}^T, \delta\boldsymbol{\omega}^T)^T$$

$$\dot{\mathbf{x}}_{\text{ins}} = \begin{pmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \partial\mathbf{g}^e/\partial\mathbf{r}^e & -2\boldsymbol{\Omega}_{ie}^e & (-(\mathbf{C}_b^e\mathbf{f}^b)\times) & \mathbf{C}_b^e & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\boldsymbol{\Omega}_{ie}^e & \mathbf{0} & \mathbf{C}_b^e \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{x}_{\text{ins}} + \mathbf{q}_{\text{ins}}$$

$$\mathbf{r}^e = \tilde{\mathbf{r}}^e - \delta\mathbf{r},\ \mathbf{v}^e = \tilde{\mathbf{v}}^e - \delta\mathbf{v}$$

$$\mathbf{f}^b = \tilde{\mathbf{f}}^b - \delta\mathbf{f},\ \boldsymbol{\omega}_{ib}^b = \tilde{\boldsymbol{\omega}}_{ib}^b - \delta\boldsymbol{\omega}$$

$$\mathbf{C}_b^e = (\mathbf{I} - (\delta\boldsymbol{\psi}\times))\tilde{\mathbf{C}}_b^e$$



- $\delta\mathbf{r}$: INS position error in e-frame (m)
- $\delta\mathbf{v}$: INS velocity error in e-frame (m/s)
- $\delta\boldsymbol{\psi}$: INS attitude error in e-frame (rad)
- $\delta\mathbf{f}$: Accelerometer bias (m/s^2)
- $\delta\boldsymbol{\omega}$: Gyro bias (rad/s)
- $\tilde{\mathbf{r}}^e$: INS position in e-frame (m)
- $\tilde{\mathbf{v}}^e$: INS velocity in e-frame (m/s)
- $\tilde{\mathbf{C}}_b^e$: INS attitude in e-frame
- $\tilde{\boldsymbol{\omega}}_{ib}^b$: INS angular rate measurement (rad/s)
- $\tilde{\mathbf{f}}^b$: INS specific force measurement (m/s^2)
- $\mathbf{r}^e$: Corrected position in e-frame (m)
- $\mathbf{v}^e$: Corrected velocity in e-frame (m/s)
- $\mathbf{C}_b^e$: Corrected attitude in e-frame
- $\boldsymbol{\omega}_{ib}^b$: Corrected angular rate (rad/s)
- $\mathbf{f}^b$: Corrected specific force (m/s^2)
- σ_a : Std-dev of accelerometer noise (m/s^2)
- σ_g : Std-dev of gyro noise (rad/s)
- σ_{ab} : Std-dev of accelerometer bias (m/s^2)
- σ_{gb} : Std-dev of gyro bias (rad/s)
- τ : Update interval of EKF (s)

State Transition and Process Noise Matrix

$$\boldsymbol{\Phi}_{\text{ins}} \approx \begin{pmatrix} \mathbf{I} & \mathbf{I}\tau & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \partial\mathbf{g}^e/\partial\mathbf{r}^e\tau & \mathbf{I} - 2\boldsymbol{\Omega}_{ie}^e\tau & (-(\mathbf{C}_b^e\mathbf{f}^b)\times)\tau & \mathbf{C}_b^e\tau & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} - \boldsymbol{\Omega}_{ie}^e\tau & \mathbf{0} & \mathbf{C}_b^e\tau \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{pmatrix}$$ (1st-order approx.) *

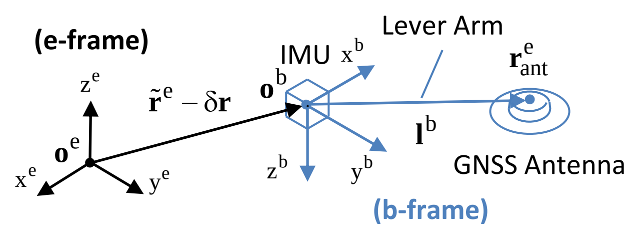
$$\mathbf{Q}_{\text{ins}} \approx \text{blkdiag}\left(\mathbf{0}, \sigma_a^2\mathbf{I}, \sigma_g^2\mathbf{I}, \sigma_{ab}^2\mathbf{I}, \sigma_{gb}^2\mathbf{I}\right)\tau$$

* To minimize the integration error in high-dynamic environment with fast attitude change, high-order approximation or finer step integration is desirable to generate a precise state transition matrix.

INS/GNSS Integration (2/3)

Loosely Coupled Integration in e-frame

$$\mathbf{x} = \mathbf{x}_{\mathrm{ins}} = (\delta\mathbf{r}^{\mathrm{T}}, \delta\mathbf{v}^{\mathrm{T}}, \delta\boldsymbol{\psi}^{\mathrm{T}}, \delta\mathbf{f}^{\mathrm{T}}, \delta\boldsymbol{\omega}^{\mathrm{T}})^{\mathrm{T}}$$



Time Update of EKF

$$\hat{\mathbf{x}}_k^- = \boldsymbol{\Phi}_k \hat{\mathbf{x}}_{k-1}^+, \ \mathbf{P}_k^- = \boldsymbol{\Phi}_k \mathbf{P}_{k-1}^+ \boldsymbol{\Phi}_k^{\mathrm{T}} + \mathbf{Q}_k \ \left(\boldsymbol{\Phi}_k = \boldsymbol{\Phi}_{\mathrm{ins}}, \ \mathbf{Q}_k = \mathbf{Q}_{\mathrm{ins}}\right)$$

Measurement Update of EKF

$$\mathbf{K}_k = \mathbf{P}_k^- \mathbf{H}_k^{\mathrm{T}} (\mathbf{H}_k \mathbf{P}_k^- \mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k)^{-1}$$

$$\hat{\mathbf{x}}_k^+ = \hat{\mathbf{x}}_k^- + \mathbf{K}_k(\mathbf{y}_k - \mathbf{h}(\hat{\mathbf{x}}_k^-)), \mathbf{P}_k^+ = (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k)\mathbf{P}_k^-$$

$$\left\{\begin{array}{l}
\mathbf{y}_k = \begin{pmatrix} \mathbf{r}_{\mathrm{gnss}} \\ \mathbf{v}_{\mathrm{gnss}} \end{pmatrix}, \mathbf{R}_k = \begin{pmatrix} \mathbf{Q}_{\mathbf{r}} & \\ & \mathbf{Q}_{\mathbf{v}} \end{pmatrix} \\
\mathbf{h}(\mathbf{x}) = \begin{pmatrix} \mathbf{r}^{e}_{\mathrm{ant}} \\ \mathbf{v}^{e}_{\mathrm{ant}} \end{pmatrix}, \mathbf{H}_k = \begin{pmatrix} -\mathbf{I} & \mathbf{0} & (\delta\mathbf{r}_{\mathrm{ant}}\times) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} & (\delta\mathbf{v}_{\mathrm{ant}}\times) & \mathbf{0} & \mathbf{C}^{e}_{b}(\mathbf{l}^{b}\times) \end{pmatrix} \\
\qquad \mathbf{C}^{e}_{b} = (\mathbf{I} - (\delta\boldsymbol{\psi}\times))\tilde{\mathbf{C}}^{e}_{b} \\
\qquad \delta\mathbf{r}_{\mathrm{ant}} = \mathbf{C}^{e}_{b}\mathbf{l}^{b} \\
\qquad \delta\mathbf{v}_{\mathrm{ant}} \approx \mathbf{C}^{e}_{b}((\tilde{\boldsymbol{\omega}}^{b}_{ib} - \delta\boldsymbol{\omega}) \times \mathbf{l}^{b}) \\
\qquad \mathbf{r}^{e}_{\mathrm{ant}} = \tilde{\mathbf{r}}^{e} - \delta\mathbf{r} + \delta\mathbf{r}_{\mathrm{ant}} \\
\qquad \mathbf{v}^{e}_{\mathrm{ant}} = \tilde{\mathbf{v}}^{e} - \delta\mathbf{v} + \delta\mathbf{v}_{\mathrm{ant}}
\end{array}\right.$$

- $\tilde{\mathbf{r}}^{e}$: INS position in e-frame (m)
- $\tilde{\mathbf{v}}^{e}$: INS velocity in e-frame (m/s)
- $\tilde{\mathbf{C}}^{e}_{b}$: INS attitude in e-frame
- $\tilde{\boldsymbol{\omega}}^{b}_{ib}$: INS angular rate measurement (rad/s)
- $\tilde{\mathbf{f}}^{b}$: INS specific force measurement (m/s^2)
- $\delta\mathbf{r}$: INS position error in e-frame (m)
- $\delta\mathbf{v}$: INS velocity error in e-frame (m/s)
- $\delta\boldsymbol{\psi}$: INS attitude error in e-frame (rad)
- $\delta\mathbf{f}$: Accelerometer bias (m/s^2)
- $\delta\boldsymbol{\omega}$: Gyro bias (rad/s)
- $\mathbf{l}^{b}$: Lever-arm vector in b-frame (m)
- $\mathbf{r}^{e}_{\mathrm{ant}}$: Antenna position in e-frame (m)
- $\mathbf{v}^{e}_{\mathrm{ant}}$: Antenna velocity in e-frame (m/s)
- $\mathbf{r}_{\mathrm{gnss}}$: GNSS position solution (ECEF) (m)
- $\mathbf{v}_{\mathrm{gnss}}$: GNSS velocity solution (ECEF) (m/s)
- $\mathbf{Q}_{\mathbf{r}}$: VC-matrix of GNSS position solution
- $\mathbf{Q}_{\mathbf{v}}$: VC-matrix of GNSS velocity solution

INS/GNSS Integration (3/3)

Tightly Coupled Integration in e-frame

$$\mathbf{x} = (\mathbf{x}_{ins}{}^T, cdt_r, c\dot{dt}_r)^T = (\delta\mathbf{r}^T, \delta\mathbf{v}^T, \delta\boldsymbol{\psi}^T, \delta\mathbf{f}^T, \delta\boldsymbol{\omega}^T, cdt_r, c\dot{dt}_r)^T$$

Time Update of EKF

$$\hat{\mathbf{x}}_k^- = \boldsymbol{\Phi}_k \hat{\mathbf{x}}_{k-1}^+, \;\; \mathbf{P}_k^- = \boldsymbol{\Phi}_k \mathbf{P}_{k-1}^+ \boldsymbol{\Phi}_k^T + \mathbf{Q}_k \quad \left(\boldsymbol{\Phi}_k = \begin{pmatrix} \boldsymbol{\Phi}_{ins} & & \\ & 1 & \\ & & 1 \end{pmatrix}, \mathbf{Q}_k = \begin{pmatrix} \mathbf{Q}_{ins} & & \\ & \infty & \\ & & \infty \end{pmatrix} \right)$$

Measurement Update of EKF

$$\mathbf{K}_k = \mathbf{P}_k^- \mathbf{H}_k^T (\mathbf{H}_k \mathbf{P}_k^- \mathbf{H}_k^T + \mathbf{R}_k)^{-1}$$

$$\hat{\mathbf{x}}_k^+ = \hat{\mathbf{x}}_k^- + \mathbf{K}_k(\mathbf{y}_k - \mathbf{h}(\hat{\mathbf{x}}_k^-)), \mathbf{P}_k^+ = (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k)\mathbf{P}_k^-$$

$$\left\{ \begin{array}{l} \mathbf{y}_k = (P_r^1, P_r^2, \ldots, P_r^m, -\lambda D_r^1, -\lambda D_r^2, \ldots, -\lambda D_r^m)^T \\ \mathbf{R}_k = \mathrm{diag}\left(\sigma_P^2, \sigma_P^2, \ldots, \sigma_P^2, \sigma_D^2, \sigma_D^2, \ldots, \sigma_D^2\right) \\ \mathbf{h}(\mathbf{x}) = \begin{pmatrix} \rho_r^1 + cdt_r - cdT^1(t^1) + I_r^1 + T_r^1 \\ \rho_r^2 + cdt_r - cdT^2(t^2) + I_r^2 + T_r^2 \\ \vdots \\ \rho_r^m + cdt_r - cdT^m(t^m) + I_r^m + T_r^m \\ \dot{\rho}_r^1 + c\dot{dt}_r - cd\dot{T}^1(t^1) \\ \dot{\rho}_r^2 + c\dot{dt}_r - cd\dot{T}^2(t^2) \\ \vdots \\ \dot{\rho}_r^m + c\dot{dt}_r - cd\dot{T}^m(t^m) \end{pmatrix}, \;\; \mathbf{H}_k = \begin{pmatrix} -\mathbf{e}_r^{1T} & \mathbf{0} & -\mathbf{e}_r^{1T}(\delta\mathbf{r}_{ant}\times) & \mathbf{0} & \mathbf{0} & 1 & 0 \\ -\mathbf{e}_r^{2T} & \mathbf{0} & -\mathbf{e}_r^{2T}(\delta\mathbf{r}_{ant}\times) & \mathbf{0} & \mathbf{0} & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\mathbf{e}_r^{mT} & \mathbf{0} & -\mathbf{e}_r^{mT}(\delta\mathbf{r}_{ant}\times) & \mathbf{0} & \mathbf{0} & 1 & 0 \\ \mathbf{0} & -\mathbf{e}_r^{1T} & -\mathbf{e}_r^{1T}(\delta\mathbf{v}_{ant}\times) & \mathbf{0} & -\mathbf{e}_r^{1T}\mathbf{C}_b^e(\mathbf{l}^b\times) & 0 & 1 \\ \mathbf{0} & -\mathbf{e}_r^{2T} & -\mathbf{e}_r^{2T}(\delta\mathbf{v}_{ant}\times) & \mathbf{0} & -\mathbf{e}_r^{2T}\mathbf{C}_b^e(\mathbf{l}^b\times) & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & -\mathbf{e}_r^{mT} & -\mathbf{e}_r^{mT}(\delta\mathbf{v}_{ant}\times) & \mathbf{0} & -\mathbf{e}_r^{mT}\mathbf{C}_b^e(\mathbf{l}^b\times) & 0 & 1 \end{pmatrix} \\ \qquad \rho_r^s = \left|\mathbf{r}^s(t^s) - \mathbf{r}_{ant}^e\right| + \delta\rho, \; \dot{\rho}_r^s = \mathbf{e}_r^{sT}(\mathbf{v}^s(t^s) - \mathbf{v}_{ant}^e) + \delta\dot{\rho} \end{array} \right.$$

dt_r : GNSS receiver clock bias (m)
$\dot{dt}_r$: GNSS receiver clock drift (m/s)
P_r^s : GNSS pseudorange (m)
D_r^s : GNSS Doppler frequency (Hz)
λ : GNSS carrier wavelength (m/cyc)
σ_P : Std-dev of pseudorange noise (m)
σ_D : Std-dev of Doppler noise (m/s)
ρ_r^s : Geometric range (m)
$\dot{\rho}_r^s$: Range rate (m/s)
$\mathbf{r}^s(t^s), \mathbf{v}^s(t^s)$: GNSS satellite position/velocity (m, m/s)
$dT^s(t^s), d\dot{T}^s(t^s)$: GNSS satellite clock bias/drift (s, s/s)
I_r^s : Ionosphere model (m)
T_r^s : Troposphere model (m)
$\mathbf{e}_r^s$: LOS (line-of-sight) vector
$\delta\rho, \delta\dot{\rho}$: Sagnac effect corrections (m, m/s)

Note: GNSS carrier phase for RTK or PPP can be involved to obtain precise GNSS receiver positions. In this case, additional parameters like phase biases and troposphere terms should be estimated as the EKF states.

Integrated INS/GNSS Performance

Applanix POS LV [1]

PCS (POS computer system) + IMU +
DMI (Distance measurement instrument) +
GNSS (Trimble BD960 and Zephyr II antenna)
Price range: $100,000 ~ $200,000

INERTIAL MEASUREMENT UNIT (IMU) [2]

Type	Operational Temperature °C	Models Used In	Maximum Data Rate	Dimensions (L x W x H) mm	Weight kg
IIMU-7[1]	-54 to +71	POSLV 420	200 Hz	158 x 158 x 124	2.5
IMU-17[1]	-40 to +60	POSLV 210/220	100 Hz	158 x 158 x 124	2.5
IMU-21[1]	-40 to +60	POSLV 610/620	200 Hz	213 x 172 x 172	4.8
IMU-42[2]	-20 to +55	POSLV 210/220	200 Hz	158 x 158 x 124	2.6
IMU-80[2]	-20 to +55	POSLV 510/520	200 Hz	161 x 120 x 126	1.9
IMU-57[2]	-20 to +55	POSLV 610/620	200 Hz	179 x 126 x 127	2.6
IMU-64[2]	-20 to +55	POSLV 420	200 Hz	158 x 158 x 124	2.6
IMU-82[2]	-40 to +65	POSLV 210/220	200 Hz	158 x 158 x 124	2.3

PERFORMANCE SPECIFICATIONS - GNSS OUTAGE, 60 SECONDS* [2]

POS LV	220 PP	220 IARTK	220 DGPS	420 PP	420 IARTK	420 DGPS	510/520 PP	510/520 IARTK	510/520 DGPS	610/620 PP	610/620 IARTK	610/620 DGPS
X,Y Position (m)	0.240	0.690	0.880	0.120	0.340	0.450	0.100	0.300	0.420	0.100	0.280	0.410
Z Position (m)	0.130	0.350	0.610	0.100	0.270	0.560	0.070	0.100	0.530	0.070	0.100	0.510
Roll & Pitch (deg)	0.060	0.060	0.060	0.020	0.020	0.020	0.005	0.008	0.008	0.005	0.005	0.005
True Heading (deg)	0.030	0.070	0.070	0.020	0.030	0.030	0.015	0.020	0.020	0.015	0.020	0.020

* All accuracy values given as RMS. Assumes typical road vehicle dynamics for initialization.

[1] https://www.applanix.com/products/poslv.htm
[2] Applanix, POS LV - designed for integration, built for performance - Data Sheet, 2017

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