For JAXA R&D

PPP - Models, Algorithms and Implementations (6-2)

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PPP - Models, Algorithms and Implementation

1. 2019-10-04 **PPP models**
   geometric range, ionosphere, troposphere, antenna PCV, earth tides, wind-up, relativity, biases, coordinates
2. 2019-10-18 **PPP algorithms**
   SPP, LSQ, GN, EKF, noise-model, RAIM/QC, LAPACK/BLAS
3. 2019-11-01 **PPP data handling**
   LC, interpolation, slip detection, RINEX, SP3, ANTEX, RTCM, CSSR
4. 2019-11-22 **PPP-AR**
   UPD/FCB, EWL/WL/NL, ILS, LAMBDA, TCAR, PAR, validation
5. 2019-12-06 **INS integration**
   INS sensors, Inertial navigation, INS integration
6. 2019-12-20 **POD of satellites**
   orbit element, orbit model, reduced-dynamic, ECI-ECEF transformation, precession/nutation, EOP

(1.5 h / session)

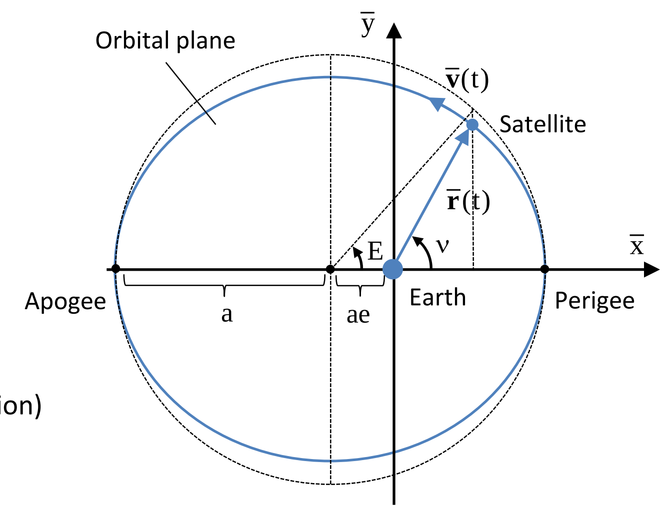
Satellite Orbit Model

Two-Body Problem

Satellite Dynamics by Two-body Problem

$$\frac{d^2\mathbf{r}(t)}{dt^2} = -\frac{GM_e\mathbf{r}(t)}{r^3}$$

Satellite Position and Velocity (in ECI)



$\mathbf{E}(t_0) = (a, e, i, \Omega, \omega, M_0)$: Orbital elements

$n = \sqrt{GM_e / a^3}$: Mean motion (rad/s)

$M = M_0 + n(t - t_0)$: Mean anomaly (rad)

$E - e\sin E = M$: Eccentric anomaly (rad) (Kepler's Equation)

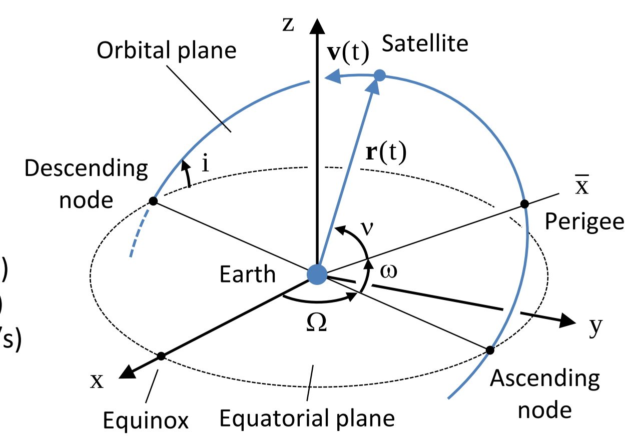
$\nu = \mathrm{ATAN2}(\sqrt{1-e^2}\sin E, \cos E - e)$: True anomaly (rad)

$r = a(1 - e\cos E)$

$$\bar{\mathbf{r}}(t) = r\begin{pmatrix}\cos\nu \\ \sin\nu\end{pmatrix}, \bar{\mathbf{v}}(t) = \frac{na^2}{r}\begin{pmatrix}-\sin E \\ \sqrt{1-e^2}\cos E\end{pmatrix}$$

$$\mathbf{R} = \mathbf{R}_Z(-\Omega)\mathbf{R}_X(-i)\mathbf{R}_Z(-\omega)$$

$$\mathbf{r}(t) = \mathbf{R}\begin{pmatrix}\bar{\mathbf{r}}(t) \\ 0\end{pmatrix}, \mathbf{v}(t) = \mathbf{R}\begin{pmatrix}\bar{\mathbf{v}}(t) \\ 0\end{pmatrix}$$



a : Semi-major axis (m)
e : Eccentricity
i : Inclination (rad)
Ω : RAAN (rad)
ω : Argument of perigee (rad)
M_0 : Mean anomaly at t_0 (rad)

GM_e : Earth's gravitational constant (m^3/s^2)
$\bar{\mathbf{r}}(t)$: Satellite position in orbital plane (m)
$\bar{\mathbf{v}}(t)$: Satellite velocity in orbital plane (m/s)
$\mathbf{r}(t)$: Satellite position in ECI (m)
$\mathbf{v}(t)$: Satellite velocity in ECI (m/s)

(RAAN: Right ascension of ascending node)

Satellite Orbit Model

State Vector of Satellite (in ECI)

$$\mathbf{x}(t) = \begin{pmatrix} \mathbf{r}(t) \\ \mathbf{v}(t) \end{pmatrix} = (x, y, z, v_x, v_y, v_z)^T$$

ODE for Satellite Dynamics

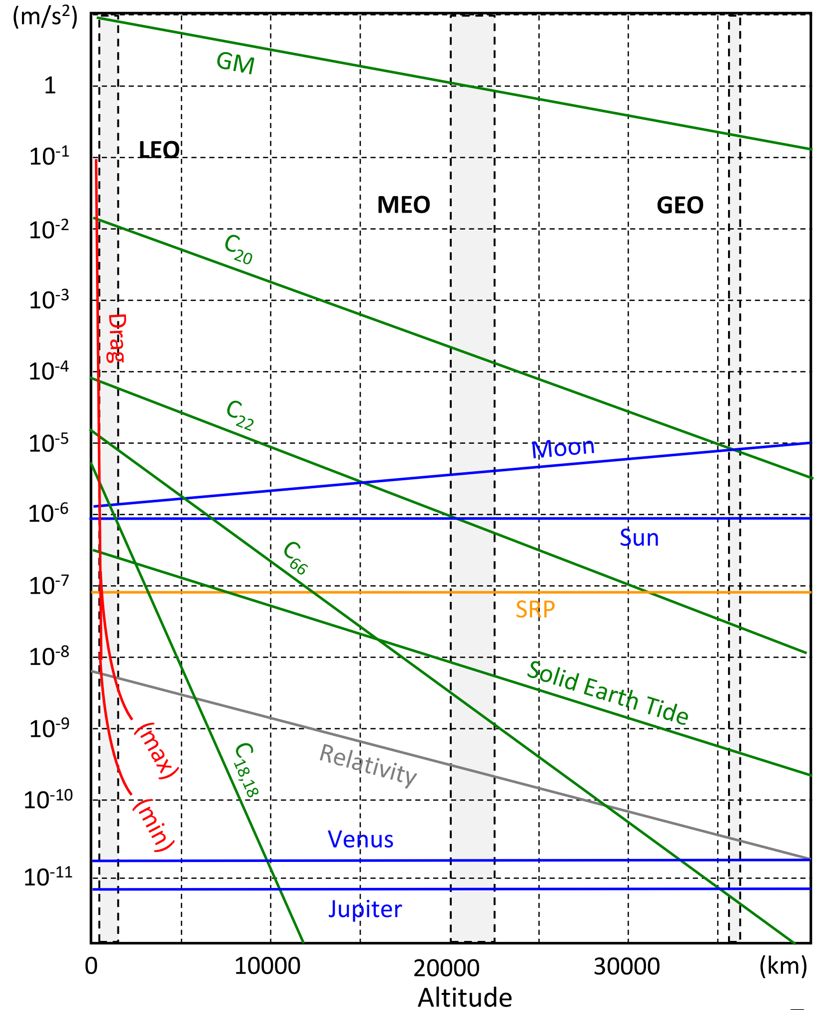
$$\frac{d\mathbf{x}(t)}{dt} = \frac{d}{dt}\begin{pmatrix} \mathbf{r}(t) \\ \mathbf{v}(t) \end{pmatrix} = \begin{pmatrix} \mathbf{v}(t) \\ \mathbf{a}(t, \mathbf{r}(t), \mathbf{v}(t), \mathbf{p}) \end{pmatrix}$$

Satellite Acceleration (in ECI)

$$\mathbf{a}(t, \mathbf{r}(t), \mathbf{v}(t), \mathbf{p}) = \mathbf{a}_{geop} + \mathbf{a}_{body} + \mathbf{a}_{drag} + \mathbf{a}_{srp} + \mathbf{a}_{thru} + \mathbf{a}_{rel} + \mathbf{a}_{emp}$$

$\mathbf{r}(t)$: Satellite position at time t in ECI (m)
$\mathbf{v}(t)$: Satellite velocity at time t in ECI (m/s)
$\mathbf{p}$: Satellite orbit parameters (C_D, C_R, ...)
$\mathbf{a}_{geop}$: Acceleration due to geopotential (m/s^2)
$\mathbf{a}_{body}$: Acceleration due to 3rd-body gravities (m/s^2)
$\mathbf{a}_{drag}$: Acceleration due to atmospheric drag (m/s^2)
$\mathbf{a}_{srp}$: Acceleration due to SRP (m/s^2)
$\mathbf{a}_{thru}$: Acceleration due to thrust force (m/s^2)
$\mathbf{a}_{rel}$: Acceleration due to general relativity (m/s^2)
$\mathbf{a}_{emp}$: Empirical acceleration (m/s^2)

Satellite Accelerations



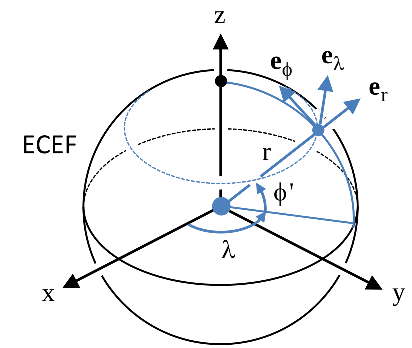
Geopotential (1/2)

Acceleration due to Geopotential

$$\mathbf{a}_{geop} = \mathbf{U}(t)^T \frac{\partial V}{\partial \mathbf{r}'} = \mathbf{U}(t)^T \left(\frac{\partial V}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial V}{\partial \phi'}\mathbf{e}_\phi + \frac{1}{r\cos\phi'}\frac{\partial V}{\partial \lambda}\mathbf{e}_\lambda \right)$$

$$V(r,\phi',\lambda) = \frac{GM_e}{r}\left\{1 + \sum_{n=2}^{\infty}\left(\frac{R_e}{r}\right)^n \sum_{m=0}^{n} \bar{P}_{nm}(\sin\phi')\left(\bar{C}_{nm}\cos m\lambda + \bar{S}_{nm}\sin m\lambda\right)\right\}$$: Geopotential

$$\frac{\partial V}{\partial \mathbf{r}'} = -\frac{GM_e}{r^2}(\mathbf{e}_r, \mathbf{e}_\phi, \mathbf{e}_\lambda)\begin{pmatrix} 1 + \sum_{n=2}^{\infty}(n+1)\left(\frac{R_e}{r}\right)^n \sum_{m=0}^{n} \bar{P}_{nm}(\sin\phi')\left(\bar{C}_{nm}\cos m\lambda + \bar{S}_{nm}\sin m\lambda\right) \\ -\cos\phi' \sum_{n=2}^{\infty}\left(\frac{R_e}{r}\right)^n \sum_{m=0}^{n} \bar{P}'_{nm}(\sin\phi')\left(\bar{C}_{nm}\cos m\lambda + \bar{S}_{nm}\sin m\lambda\right) \\ -\frac{1}{\cos\phi'}\sum_{n=2}^{\infty}\left(\frac{R_e}{r}\right)^n \sum_{m=0}^{n} \bar{P}_{nm}(\sin\phi') m\left(-\bar{C}_{nm}\sin m\lambda + \bar{S}_{nm}\cos m\lambda\right) \end{pmatrix}$$

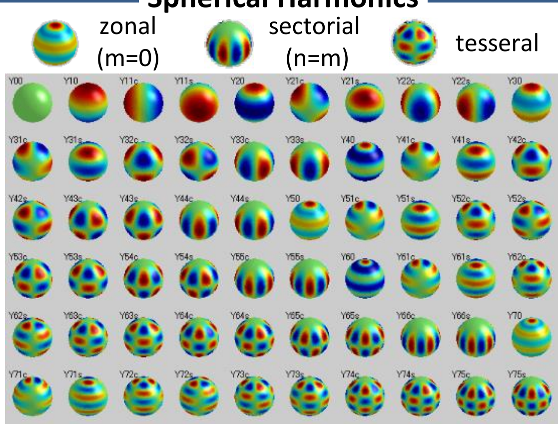


$$\mathbf{r}' = \mathbf{U}(t)\mathbf{r}(t) = (x,y,z)^T, r = |\mathbf{r}'|, \phi' = z/\sqrt{x^2+y^2}, \lambda = \mathrm{ATAN2}(y,x)$$

$$\mathbf{e}_r = \begin{pmatrix} \cos\lambda\cos\phi' \\ \sin\lambda\cos\phi' \\ \sin\phi' \end{pmatrix}, \mathbf{e}_\phi = \begin{pmatrix} -\cos\lambda\sin\phi' \\ -\sin\lambda\sin\phi' \\ \cos\phi' \end{pmatrix}, \mathbf{e}_\lambda = \begin{pmatrix} -\sin\lambda \\ \cos\lambda \\ 0 \end{pmatrix}$$

$\mathbf{r}(t)$: Satellite Position in ECI (m) $\mathbf{r}'$: Satellite Position in ECEF (m) $\mathbf{U}(t)$: ECI to ECEF trans. matrix R_e : Equatorial radius of the Earth (m)

Spherical Harmonics



$$Y_{n0} = P_{n0}(\sin\phi'), Y_{nmc} = P_{nm}(\sin\phi')\cos m\lambda, Y_{nms} = P_{nm}(\sin\phi')\sin m\lambda$$

Associated Legendre Functions

$$P_{00}(x) = 1,\ P_{10}(x) = x, P_{n-1,n}(x) = 0, P'_{00}(x) = 0$$

$$P_{nn}(x) = (2n-1)\sqrt{1-x^2}P_{n-1,n-1}(x) \quad (1)$$

$$P_{nm}(x) = \frac{(2n-1)xP_{n-1,m}(x) - (n+m-1)P_{n-2,m}(x)}{n-m} \quad (2)$$

$$P'_{nm}(x) = \frac{nP_{nm}(x) - (n+m)P_{n-1,m}(x)}{x^2-1}$$

$$N_{nm} = \begin{cases} \sqrt{2n+1} & (m=0) \\ \sqrt{\frac{2(2n+1)(n-m)!}{(n+m)!}} & (m>0) \end{cases}$$

$$\bar{P}_{nm} = N_{nm}P_{nm}, \bar{P}'_{nm} = N_{nm}P'_{nm}$$

Recursive Algorithm

n \ m	0	1	2	3	4	5
0	P_{00}	0	0	0	0	0
1	P_{10}	P_{11}	0	0	0	0
2	P_{20}	P_{21}	P_{22}	0	0	0
3	P_{30}	P_{31}	P_{32}	P_{33}	0	0
4	P_{40}	P_{41}	P_{42}	P_{43}	P_{44}	0
5	P_{50}	P_{51}	P_{52}	P_{53}	P_{54}	P_{55}

Geopotential (2/2)

Geopotential Models

Model	Author	Coefficients	Tide System	References
JGM-3	NASA GSFC, UT CSR	70 x 70	zero-tide	Tapley et al., 1996
EGM96	NASA GSFC, NIMA	360 x 360	tide-free	https://earth-info.nga.mil/GandG/wgs84/gravitymod/egm96/egm96.html
EIGEN-GL04C	GFZ, GRGS	360 x 360	tide-free	http://op.gfz-potsdam.de/grace/results/grav/g005_eigen-gl04c.html
EGM2008 [1]	NGA	2190 x 2190	tide-free	https://earth-info.nga.mil/GandG/wgs84/gravitymod/egm2008

Corrections to Geopotential Coefficients

$$\bar{C}_{20} = \bar{C}_{20,\mathrm{model}} + \dot{\bar{C}}_{20}(t - t_{J2000}) + \Delta\bar{C}_{20}$$

$$\bar{C}_{30} = \bar{C}_{30,\mathrm{model}} + \dot{\bar{C}}_{30}(t - t_{J2000}) + \Delta\bar{C}_{30}$$

$$\bar{C}_{40} = \bar{C}_{40,\mathrm{model}} + \dot{\bar{C}}_{40}(t - t_{J2000}) + \Delta\bar{C}_{40}$$

(J2000 = 2000-01-01 12:00 TT)

$$\bar{C}_{21} = \sqrt{3}\bar{x}_p\bar{C}_{20} - \bar{x}_p\bar{C}_{22,\mathrm{model}} + \bar{y}_p\bar{S}_{22,\mathrm{model}} + \Delta\bar{C}_{21}$$

$$\bar{S}_{21} = -\sqrt{3}\bar{y}_p\bar{C}_{20} - \bar{y}_p\bar{C}_{22,\mathrm{model}} - \bar{x}_p\bar{S}_{22,\mathrm{model}} + \Delta\bar{S}_{21}$$

$$\bar{C}_{nm} = \bar{C}_{nm,\mathrm{model}} + \Delta\bar{C}_{nm}, \bar{S}_{nm} = \bar{S}_{nm,\mathrm{model}} + \Delta\bar{S}_{nm} \quad \text{(others)}$$

$\bar{C}_{nm,\mathrm{model}}, \bar{S}_{nm,\mathrm{model}}$: Geopotential model coefficients

$\dot{\bar{C}}_{nm}$: Rate of geopotential coefficients (year^{-1}) ([1] 6.1)

$\bar{x}_p, \bar{y}_p$: IERS conventional mean pole (rad) ([1] 7.1.4)

$\Delta\bar{C}_{nm}, \Delta\bar{S}_{nm}$: Corrections to geopotential coefficients ([1] 6.2 ~ 6.5)

(solid earth tides + ocean tides + solid earth pole tide + ocean pole tide)

EGM2008 Model

$(n, m \le 5)$

```
GM_e = 398600.4415 (km^3/s^2)
R_e  = 6378136.3 (m)
n  m          C_nm                   S_nm
2  0  -0.484165143790815E-03   0.000000000000000E+00
2  1  -0.206615509074176E-09   0.138441389137979E-08
2  2   0.243938357328313E-05  -0.140027370385934E-05
3  0   0.957161207093473E-06   0.000000000000000E+00
3  1   0.203046201047864E-05   0.248200415856872E-06
3  2   0.904787894809528E-06  -0.619005475177618E-06
3  3   0.721321757121568E-06   0.141434926192941E-05
4  0   0.539965866638991E-06   0.000000000000000E+00
4  1  -0.536157389388867E-06  -0.473567346518086E-06
4  2   0.350501623962649E-06   0.662480026275829E-06
4  3   0.990856766672321E-06  -0.200956723567452E-06
4  4  -0.188519633023033E-06   0.308803882149194E-06
5  0   0.686702913736681E-07   0.000000000000000E+00
5  1  -0.629211923042529E-07  -0.943698073395769E-07
5  2   0.652078043176164E-06  -0.323353192540522E-06
5  3  -0.451847152328843E-06  -0.214955408306046E-06
5  4  -0.295328761175629E-06   0.498070550102351E-07
5  5   0.174811795496002E-06  -0.669379935180165E-06
```

[1] G. Petit and B. Luzum (eds.), IERS Technical note No. 36: IERS Conventions (2010), 2010

3rd-body Gravities

Acceleration due to 3rd-body Gravities

$$\mathbf{a}_{body} = \sum_p \left\{ GM_p \left(\frac{\mathbf{r}_p(t) - \mathbf{r}(t)}{\left|\mathbf{r}_p(t) - \mathbf{r}(t)\right|^3} - \frac{\mathbf{r}_p(t)}{\left|\mathbf{r}_p(t)\right|^3} \right) \right\}$$

$\mathbf{r}(t)$: Satellite position at time t in ECI (m)
$\mathbf{r}_p(t)$: Planet p position at time t in ECI (m)
GM_p : Planet p gravitational constants (m^3/s^2)

Planet Positions by Planetary Ephemeris

NASA JPL DE (Planetary and Lunar Ephemerides) [1]

Ephemeris	Release	Span (AD)	Time Scale	Coordinates Frame	Additional Parameters
DE200	1981	Dec 9, 1599 - Mar 31, 2169	TDB	Dynamical equator and equinox in J2000	Nutation
DE405	1998	Dec 9, 1599 - Feb 20, 2201	TDB	ICRF	Nutation, libration
DE421	2008	Jul 29, 1899 - Oct 9, 2053	TDB	ICRF	Nutation, libration

$$\tau = 2(t - t_k)/(t_{k+1} - t_k) - 1 \qquad (t_k \le t < t_{k+1})$$

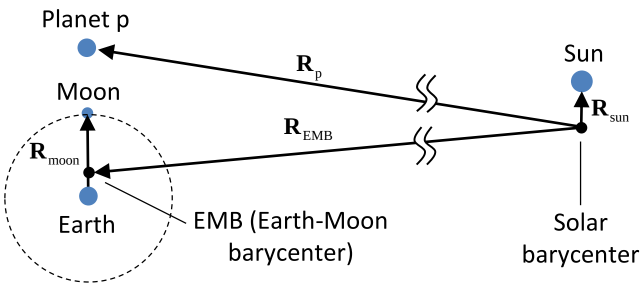
$$\mathbf{R}_p(t) = \sum_{i=0}^{n-1} \mathbf{a}_{p,i} T_i(\tau)$$

$$\mathbf{r}_{sun}(t) = \mathbf{R}_{sun}(t) - \mathbf{R}_{EMB}(t) + \mathbf{R}_{moon}(t)/(1+\mu)$$

$$\mathbf{r}_{moon}(t) = \mathbf{R}_{moon}(t)$$

$$\mathbf{r}_p(t) = \mathbf{R}_p(t) - \mathbf{R}_{sun}(t) + \mathbf{r}_{sun}(t) \qquad \text{(except for sun and moon)}$$

$\mathbf{R}_p(t)$: Planet p ephemeris position at time t (m)
$\mathbf{a}_{p,i}$: Planet p i th-order coefficients in ephemeris
μ : Ratio of Earth's / Moon's masses
$T_i(\tau)$: Chebyshev polynomial $(T_0(\tau) = 0, T_1(\tau) = 1, T_i(\tau) = 2\tau T_{i-1}(\tau) - T_{i-2}(\tau))$



p = { 1: Mercury, 2: Venus, 3: EMB, 4: Mars,
5: Jupiter, 6: Saturn, 7: Uranus,
8: Neptune, 9: Pluto, 10: Moon, 11: Sun

[1] https://ssd.jpl.nasa.gov/?planet_eph_export, ftp://ssd.jpl.nasa.gov/pub/eph/planets/planets

Atmospheric Drag

Acceleration due to Atmospheric Drag

$$\mathbf{a}_{drag} = -\frac{1}{2} C_D \frac{A}{m} \rho(\mathbf{r}(t)) |\mathbf{v}_r| \mathbf{v}_r$$

$$\mathbf{v}_r = \mathbf{v}(t) - \boldsymbol{\omega}_e \times \mathbf{r}(t)$$

C_D : Drag coefficient
A : Cross-section area of satellite (m^2)
m : Mass of satellite (kg)
$\rho(\mathbf{r}(t))$: Total atmospheric density (kg/m^3)
$\mathbf{v}_r$: Relative velocity of satellite to atmosphere (m/s)
$\boldsymbol{\omega}_e$: Earth rotation vector

Upper Atmosphere Models

Model	Release	Author	Altitude	S/W Library	References
Jacchia 1977	1977	L. G. Jacchia	90 - 250 km	Fortran	L. G. Jacchia, 1977
NRLMSISE-00	2000	US NRL	0 - 500 km	Fortran, C	https://www.nrl.navy.mil/ssd/branches/7630/modeling-upper-atmosphere
JB2008	2008	B. R. Bowman et al.	175 - 1000 km	Fortran	http://sol.spacenvironment.net/jb2008/
DTM-2013	2014	S. Bruinsma	200 - 900 km	?	S. Bruinsma, 2015

Solar Activity Data for Atmosphere Models [1][2]

K-index : Local index of the 3-hours range in magnetic activity relative to an assumed quiet-day curve
Kp-index : 3-hours range index of mean K-index from 13 geomagnetic observatories sites.
Ap-index : 3-hours range index derived from Kp index
F10.7-index : Solar radio flux at 10.7 cm (2800 MHz) radio wave

[1] Celestrak Space Weather Data (https://www.celestrak.com/SpaceData/)
[2] Celestrak Space Weather Data Documentation (https://www.celestrak.com/SpaceData/SpaceWx-format.php)

SRP (1/2)

Acceleration due to SRP (Solar Radiation Pressure)

(1) Cannonball Model

$$\mathbf{a}_{srp} = -F\left(\frac{AU}{R}\right)^2 P_{sun} C_R \frac{A}{m}\mathbf{e}_d$$

AU : Astronomical unit (= 1.49598 x 10^{11} m)
R : Sun-satellite range (m)
P_{sun} : Solar constant (=4.56 x 10^{-6} N/m^2)
C_R : Radiation pressure coefficient
A : Cross-section area of satellite (m^2)
m : Mass of satellite (kg)

(2) Macro Satellite Model

$$\mathbf{a}_{srp} = -F\left(\frac{AU}{R}\right)^2 \frac{P_{sun}}{m}\sum_s A_s\left\{\cos\theta_s\left((1-\varepsilon_s)\mathbf{e}_d + 2\varepsilon_s\cos\theta_s\mathbf{n}_s\right)\right\} \quad (\cos\theta_s = \mathbf{n}_s\cdot\mathbf{e}_d > 0)$$

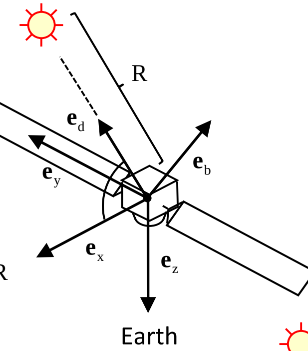
A_s : Cross-section area of surface s (m^2)
ε_s : Reflectivity of surface s
$\mathbf{n}_s$: Unit normal vector of surface s

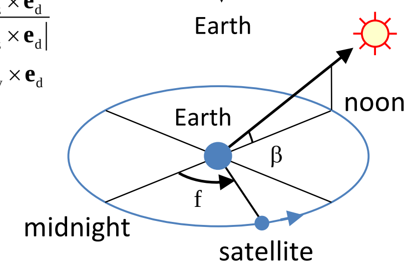
(3) Empirical Model

$$\mathbf{a}_{srp} = F\left(\frac{AU}{R}\right)^2\left((D_0 + D_c\cos f + D_s\sin f)\mathbf{e}_d + (B_0 + B_c\cos f + B_s\sin f)\mathbf{e}_b + (Y_0 + Y_c\cos f + Y_s\sin f)\mathbf{e}_y\right)$$

(4) Other Accelerations: albedo, thermal radiation, ...

$$R = |\mathbf{r}_{sun} - \mathbf{r}(t)|$$
$$\mathbf{e}_z = -\mathbf{r}(t)/|\mathbf{r}(t)|$$
$$\mathbf{e}_d = (\mathbf{r}_{sun} - \mathbf{r}(t))/R$$
$$\mathbf{e}_y = \frac{\mathbf{e}_z\times\mathbf{e}_d}{|\mathbf{e}_z\times\mathbf{e}_d|}$$
$$\mathbf{e}_b = \mathbf{e}_y\times\mathbf{e}_d$$



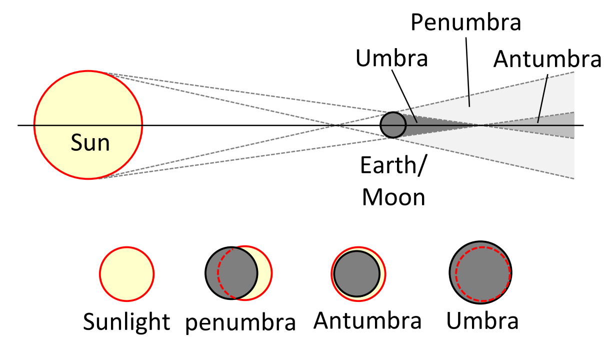


Eclipse (Shadow) Model

$$F = \begin{cases} 1 & \text{(in sunlight)} \\ 0 & \text{(in umbra)} \\ 1 - a^2/b^2 & \text{(in antumbra)} \\ 1 - (a^2\phi_1 + b^2\phi_2 - ac\sin\phi_1)/(\pi b^2) & \text{(in penumbra)} \end{cases} \quad \text{: Shadow function}$$

a : apparent radius of earth/moon (rad)
b : apparent radius of sun (rad)
c : apparent separation (rad)

$$\phi_1 = \arccos\left((c^2 + a^2 - b^2)/(2ca)\right)$$
$$\phi_2 = \arccos\left((c^2 + b^2 - a^2)/(2cb)\right)$$

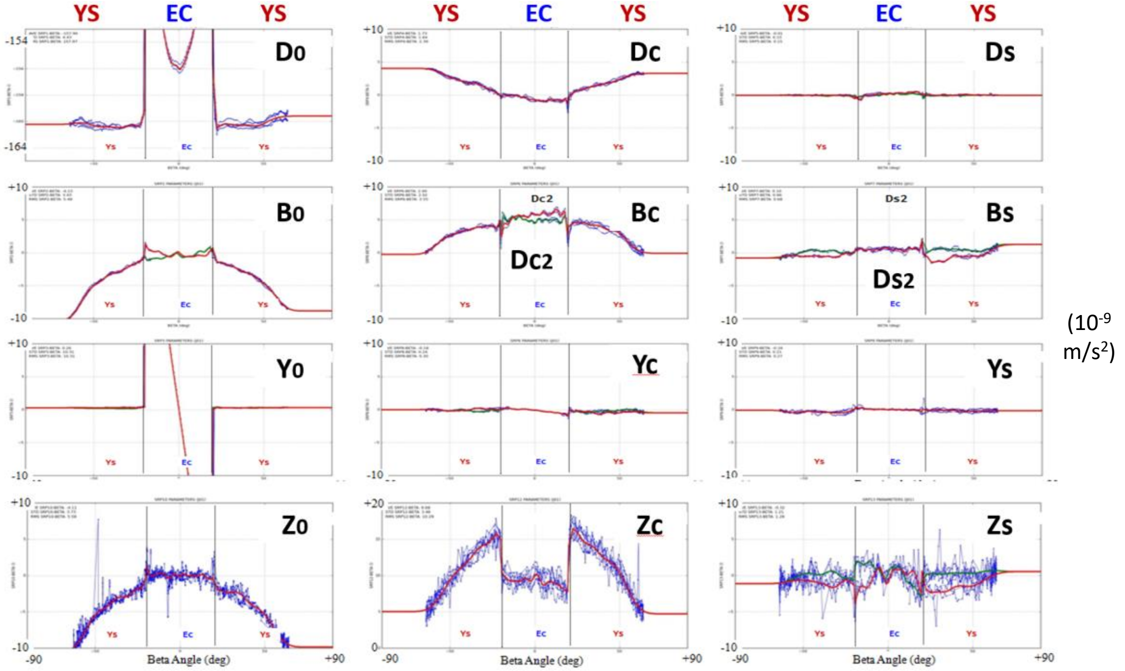


SRP (2/2)

Specific Empirical Model for QZS-1 (EDBY) [1]

$$\mathbf{a}_{srp} = F\left(\frac{AU}{R}\right)^2 \begin{cases} (D_0 + D_c\cos f + D_s\sin f)\mathbf{e}_d + (B_0 + B_c\cos f + B_s\sin f)\mathbf{e}_b + (Y_0 + Y_c\cos f + Y_s\sin f)\mathbf{e}_y + (Z_0 + Z_c\cos f + Z_s\sin f)\mathbf{e}_z & \text{(YS)} \\ (D_0 + D_c\cos f + D_s\sin f + D_{2c}\cos 2f + D_{2s}\sin 2f)\mathbf{e}_d + B_0\mathbf{e}_b + (Y_0 + Y_c\cos f + Y_s\sin f)\mathbf{e}_y + (Z_0 + Z_c\cos f + Z_s\sin f)\mathbf{e}_z & \text{(EC)} \end{cases}$$

Model Coefficients wrt Beta Angle [1]



[1] T. Takasu et al., QZSS-1 Precise Orbit Determination by MADOCA, International Symposium on GNSS 2015, Kyoto, Japan

Numerical Integrations

Solve an Initial Value Problem of ODE

$$\mathbf{y}(t_0) = \mathbf{y}_0, \frac{d\mathbf{y}(t)}{dt} = \mathbf{f}(t, \mathbf{y}(t)) \longrightarrow \mathbf{y}(t_1), \mathbf{y}(t_2), \mathbf{y}(t_3), \mathbf{y}(t_4), \ldots$$

Numerical Integration Algorithms

Runge-Kutta methods : RK4, RK8, RKF4(5), RK8(7)13M, RKN12(10)17M, ...
Multi-step methods : Adams-Bashforth, Adams-Moulton and predictor-correlator
Extrapolation methods : Bulirsch-Stoer method, Gragg's method
Direct integration of 2nd-order ODE: Runge-Kutta-Nyström, Stoermer and Cowell
Variable order/variable step methods, ...

RK4 (4th-order 4-stage Runge-Kutta)

$$\mathbf{y}(t_{k+1}) = (\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4)/6 \qquad (h = t_{k+1} - t_k,\ \delta\mathbf{y} = O(h^4))$$

$$\mathbf{k}_1 = \mathbf{f}(t_k, \mathbf{y}(t_k)), \mathbf{k}_2 = \mathbf{f}(t_k + h/2, \mathbf{y}(t_k) + h\mathbf{k}_1/2), \mathbf{k}_3 = \mathbf{f}(t_k + h/2, \mathbf{y}(t_k) + h\mathbf{k}_2/2), \mathbf{k}_4 = \mathbf{f}(t_k + h, \mathbf{y}(t_k) + h\mathbf{k}_3)$$

Orbit Propagation Error (two-body problem, ecc=0.1)

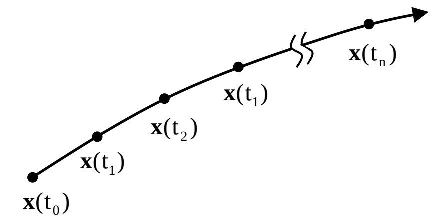
Integration Span	RK4 (step=30 s)		RK4 (step=10 s)		RK8 (step=30 s)		RKF4(5)		Gragg' method	
	pos (m)	vel (m/s)	pos (m)	vel (m/s)	pos (m)	vel (m/s)	pos (m)	vel (m/s)	pos (m)	vel (m/s)
1 day	41.497	0.0447	0.233	0.0003	0.000	0.0000	0.006	0.0000	0.000	0.0000
3 days	308.27	0.2691	1.644	0.0014	0.000	0.0000	0.006	0.0001	0.000	0.0000
7 days	1739.9	1.8868	7.509	0.0083	0.000	0.0000	0.149	0.0001	0.003	0.0000
30 days	26272.6	23.2447	112.067	0.0984	0.006	0.0000	1.460	0.0014	0.042	0.0000

Orbit Propagation

Orbit Propagation

$$\mathbf{x}(t_0) = \begin{pmatrix} \mathbf{r}(t_0) \\ \mathbf{v}(t_0) \end{pmatrix}, \quad \frac{d\mathbf{x}(t)}{dt} = \begin{pmatrix} \mathbf{v}(t) \\ \mathbf{a}(t, \mathbf{r}(t), \mathbf{v}(t), \mathbf{p}) \end{pmatrix} \xrightarrow{\text{Numerical Integration}} \mathbf{x}(t_1), \mathbf{x}(t_2), \mathbf{x}(t_3), \ldots, \mathbf{x}(t_n)$$

Initial states ODE for satellite dynamics



State Transition Matrix and Sensitivity Matrix

State transition matrix from time t_0 to time t:

$$\mathbf{\Phi}(t, t_0) = \frac{\partial \mathbf{x}(t)}{\partial \mathbf{x}(t_0)} = \begin{pmatrix} \partial \mathbf{r}(t)/\partial \mathbf{r}(t_0) & \partial \mathbf{r}(t)/\partial \mathbf{v}(t_0) \\ \partial \mathbf{v}(t)/\partial \mathbf{r}(t_0) & \partial \mathbf{v}(t)/\partial \mathbf{v}(t_0) \end{pmatrix}$$

Sensitivity matrix wrt parameters p:

$$\mathbf{S}(t) = \frac{\partial \mathbf{x}(t)}{\partial \mathbf{p}} = \begin{pmatrix} \partial \mathbf{r}(t)/\partial p_1 & \partial \mathbf{r}(t)/\partial p_2 & \cdots \\ \partial \mathbf{v}(t)/\partial p_1 & \partial \mathbf{v}(t)/\partial p_2 & \cdots \end{pmatrix}$$

(1) Numerical Integration of Variational Equation

$$\left(\mathbf{\Phi}(t_0, t_0), \mathbf{S}(t_0)\right) = (\mathbf{I}, \mathbf{0}) \quad \text{Initial matrix}$$

$$\frac{d}{dt}\left(\mathbf{\Phi}(t, t_0), \mathbf{S}(t)\right) = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ \dfrac{\partial \mathbf{a}(t, \mathbf{r}(t), \mathbf{v}(t), \mathbf{p})}{\partial \mathbf{r}(t)} & \dfrac{\partial \mathbf{a}(t, \mathbf{r}(t), \mathbf{v}(t), \mathbf{p})}{\partial \mathbf{v}(t)} \end{pmatrix} \left(\mathbf{\Phi}(t, t_0), \mathbf{S}(t)\right) + \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \dfrac{\partial \mathbf{a}(t, \mathbf{r}(t), \mathbf{v}(t), \mathbf{p})}{\partial \mathbf{p}} \end{pmatrix} \quad \text{Variational eq.}$$

$$\xrightarrow{\text{Numerical Integration}} \mathbf{\Phi}(t_1, t_0), \mathbf{\Phi}(t_2, t_0), \mathbf{\Phi}(t_3, t_0), \ldots, \mathbf{\Phi}(t_n, t_0), \mathbf{S}(t_1), \mathbf{S}(t_2), \mathbf{S}(t_3), \ldots, \mathbf{S}(t_n)$$

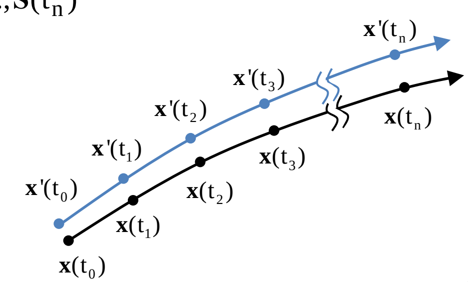
(2) Differential Quotient Approximations (Numerical Differential)

$$\mathbf{x}'(t_0) = \mathbf{x}(t_0) + (\delta x, 0, 0, \ldots, 0)^T \xrightarrow{\text{Orbit propagation}} \mathbf{x}'(t_1), \mathbf{x}'(t_2), \mathbf{x}'(t_3), \ldots, \mathbf{x}'(t_n)$$

$$\frac{\partial \mathbf{x}(t_k)}{\partial x_0} \approx \frac{\mathbf{x}'(t_k) - \mathbf{x}(t_k)}{\delta x},$$

... (for all initial states and parameters)

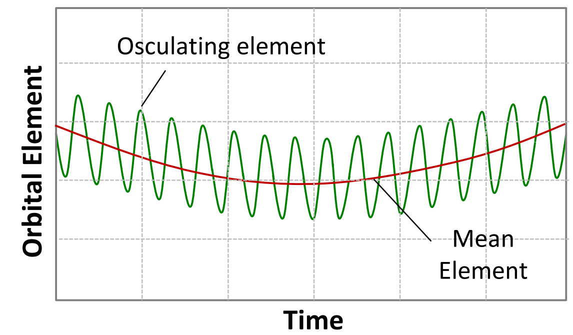
Notes: Carefully select delta-x value to minimize approximation error



Orbital Elements

Orbital Elements

- **Osculating elements (Kepler elements)**
  Instant elements derived from two-body problem with short and long period variations
- **Mean elements**
  Averaged elements in some selected time w/o short or long period variations



NORAD TLE (Two Line Elements) [1]

```
GPS BIII-1  (PRN 04)
1 43873U 18109A   20071.16240480  .00000015  00000-0  00000-0 0  9995
2 43873  55.0024 182.5846 0007249 222.9007 229.4123  2.00558681  9200
```

: Satellite name
: Line 1
: Line 2

Line 1
- Satellite catalog number
- U=Unclassified, C=Classified, S=Secret
- International Designator
- Epoch year (two digit)
- Epoch day of the year (days)
- First derivative of mean motion (revs/day^2)
- Second derivative of mean motion * (revs/day^3)
- Drag term BSTAR (1/earth radii) *
- Element set number
- Checksum

Line 2
- Satellite catalog number
- Inclination (deg)
- RAAN (deg)
- Eccentricity *
- Argument of perigee (deg)
- Mean anomaly (deg)
- Mean motion (revs/day)
- Revolution number at epoch (revs)
- Checksum

(* Leading decimal point assumed)

Coordinate System : TEME, **Time System** : UTC (no formal statement)
Orbit Propagator [2][3] : SGP, SGP4, SDP4, SGP8 or SDP8

[1] CelesTrak (https://celestrak.com), Space-Track.org (https://space0track.org)
[2] F. R. Hoots and R. L. Roehrich, Spacetrack report No.3: Models for propagation of NORAD element sets, 1988
[3] D. A. Vallado et al., Revisiting Spacetrack Report #3, American Institute of Aeronautics and Astronautics (AIAA), 2006